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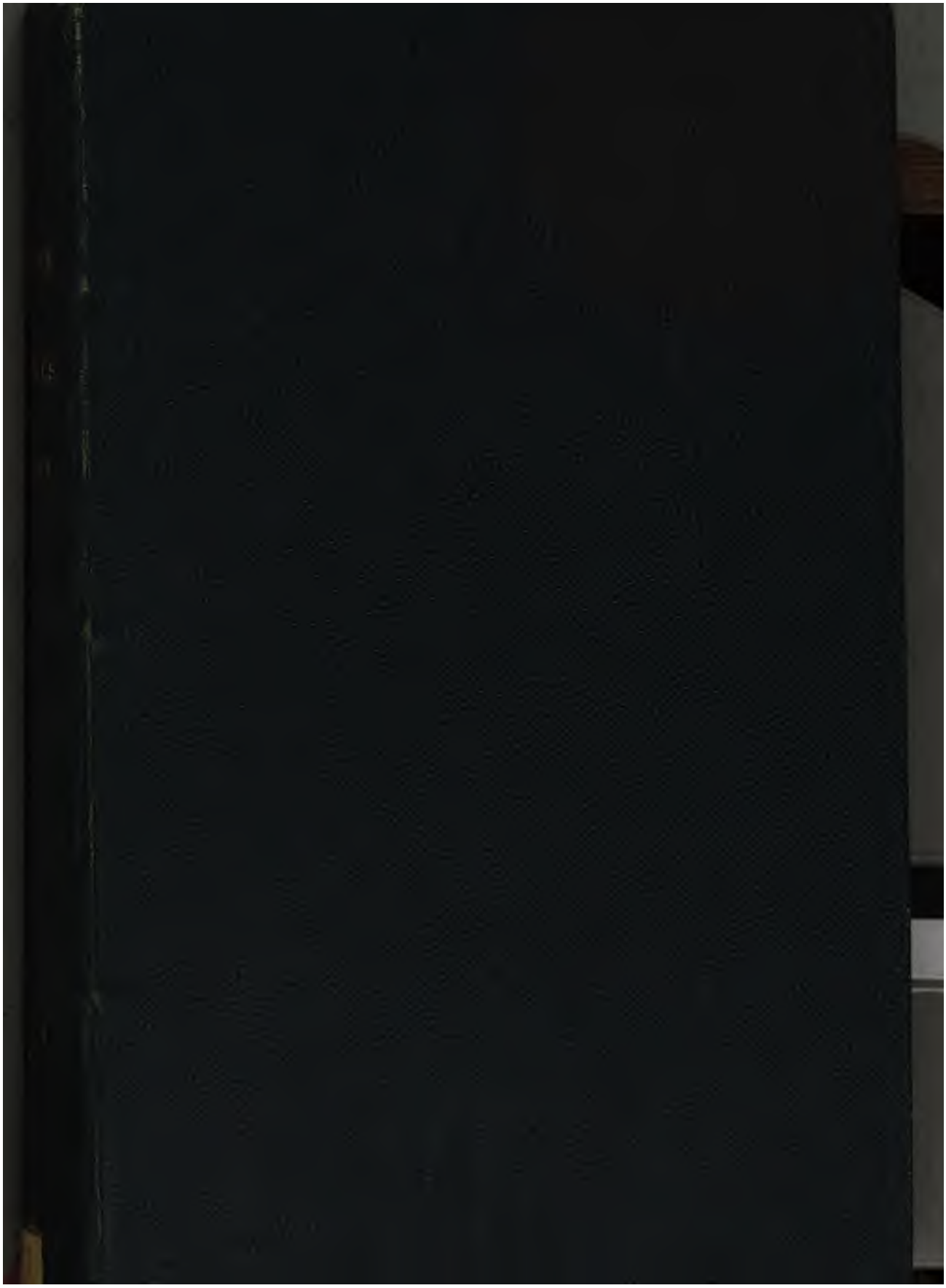
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A

TREATISE

ON

MECHANICS:

INTENDED AS AN

INTRODUCTION

TO THE STUDY OF

Natural Philosophy.

BY

THE REV. B. BRIDGE, B.D. F.R.S.

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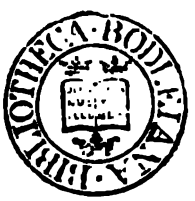
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THIS work had been published some months, when the author's attention was called to an error in the solution of the Problem at page 185, Part IV. This mistake is now corrected; and the principle of the solution of the Problem as it stands at present will be found to agree with that by which it is solved in *Simpson's Miscellaneous Tracts*, page 131, *Ed.* 1757. In the revision of the work, some further corrections and alterations have been made. The substance of the Notes, which were annexed to the end of Part II, are now introduced into the Text: and a new form has been given to the First Lecture. The work is still divided into Four Parts, according to the following arrangement.

PART I. relates to the Rectilinear Motion of Bodies both by Impulse and Gravity; the Composition and Resolution of Motion, with the Solution of the Problem for resolving any Number of Forces into the direction of Three Axes at Right Angles to each other; the Method of finding the Center of Gravity of a Body or System of Bodies; the Motion of the common Center of Gravity of a System; the Collision of Hard and Elastic Bodies; and the Motion of Projectiles.

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The first Three Lectures of Part II. contain the Doctrine of Equilibrium, as applied to the *Mechanical Powers*; the Fourth relates to the Pressure and Tension of Cords, including the Construction of the Funicular Polygon and Catenary. The last Lecture comprehends the common Theory for estimating the Strength, Stress, and Pressure of Beams.

PART III. commences with the Motion of Bodies over Inclined Planes and Pulleys; and then proceeds to the Rotatory and Vibratory Motion of Bodies about a Fixed Axis; with the Investigation of the Centers of Gyration, Oscillation, Percussion, and Spontaneous Rotation of a Body or System of Bodies. In this Part of the work will also be found some plain Theorems for ascertaining the Maximum Effects of Machinery.

PART IV. contains Three Lectures upon Miscellaneous Subjects; such as, the Descent of Bodies over Pulleys by *Variable* Forces; the Vibration of Cords; the Oscillations of Bodies in Circular and Cycloidal Arcs; the Angular Motion of Bodies suspended from the Arms of a Straight Lever; &c. &c.; and concludes with the Method of finding the actual Time and Velocity of a Falling Body, on supposition that the Force of Gravity varies inversely as the square of the distance from the Earth's center.

As this work was designed for an Elementary Treatise rather than a complete System of Mechanics, it cannot be expected to contain much that is altogether new. In its composition, the most approved Authors have been resorted to; and a clear arrangement of the subject has been one

one of the principal objects which the author has endeavoured to attain. How far he has succeeded in this respect, must be left for the Public to judge.

To those sources of information which have long been common to all Writers on this branch of science, it is superfluous for him to enumerate his particular obligations; nor is it necessary, he conceives, to specify all the more recent publications, which, it is evident, he must have consulted. It is proper however to state, that in writing Part III. he had *Atwood's Treatise on the Rectilinear Motion and Rotation of Bodies* constantly before him. That work is now become extremely scarce and dear; and he thought, that in attempting to give some of its Theorems a more elementary form, he should be rendering a service to the Mathematical Student. He has also to acknowledge, that the Theorem for the construction of the Funicular Polygon, as it stands in page 315, is little more than a translation of the same as given by *Francœur* in his *Traité Elémentaire de Méchanique:* and that the Collection of *Cambridge Problems* has furnished him with many useful and valuable Examples for the illustration of his subject.

After he had made considerable progress in printing the work, circumstances occurred which induced him to alter his original plan: this occasioned a division of it into *two* volumes of very unequal size. The

The matter is now so arranged as to be all bound up in *one*; and any possessors of the two volumes who desire to avail themselves of this arrangement, may be supplied with the extra sheets for that purpose, without expense, on application to the Publishers.

EAST-INDIA COLLEGE,
November 1814.



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ERRATA.

VOL. I.

Page 25, line 13, for AFG read gFG.
— 55, — 22, for "the subject of Projectiles will be *resumed* in the 6th Lecture;" read, "but the motion of Projectiles will *form the subject* of the 6th Lecture."
— 62, — 12, for *then*, read *and*.
— 64, — 8, for *force*, read *forces*.
— 96, — 12, for *effect*, read *affect*.
— 106, line 9, for K read R.
— 138, — 2, for *Area* pAp', read $\frac{1}{2}$ *Area* pAp'.
— 194, In the Figure belonging to this page, supply the letter P *between the letters* C *and* O.
— 197, — 1, for Lm' read $L'm'$.
— ibid. — 2, for $L'M'$, read $L'm'$.
— 204, — 4, for P read PQ.
— 223, — 17, for *sin.* 60, read *sin.* 60°.
— 227, last line, for *angle* read *axle*.
— 238, — 15, for *at* CB, read *a* Cb.
— 330, — 9, for *contract* read *construct*.
— 347, — 2, for *sustained* read *sustain*.

VOL. II.

Page 2, *Erase* "(See page 21, Part I)."
— 49, *Erase* "(See Note III. Vol. I)."
— 51, lines 17 and 19, for PA^2 read SA^2.
— 70, last line, for "*See the Note upon this subject at the end of this volume,*" read, "See the observations upon this subject, in Atwood's Treatise, Sect. VI. page 240, and Sect. IX. page 367."
— 125, last line but one, for "(*dividing by* wx^2)," read, "(*by taking the fluent, and dividing by* wx^2)."

VOL. I.

A

TREATISE

ON

MECHANICS.

Part I.

ON THE RECTILINEAR MOTION OF BODIES—COMPOSITION AND RESOLUTION OF MOTION—THE CENTER OF GRAVITY—COLLISION OF BODIES—AND PROJECTILES.

Lecture I.

ON MOTION, AND THE LAWS OF MOTION.

ALL bodies with which we are acquainted, besides those essential properties more immediately suggested to us by our senses, and inseparable from our very conceptions of matter, such as, *extension, figure, solidity, divisibility, &c.* are found by experience to possess two qualities intimately connected with the Phænomena and the Laws of Motion: these are,

GRAVITY and INERTIA.

I.

1. GRAVITY is that property, by which all the particles of matter, and all the bodies of which they are

are the elements, tend towards each other; which, in bodies existing near the Earth's surf operates as "a tendency to descend towards *center*." It is in this latter and limited sense t the term *gravity* is generally used in the follow treatise.

2. This property discovers itself, not only in motion of falling bodies, but in the pressure exe by one portion of matter upon another wl sustains it; and bodies descending freely unde influence, whatever be their figure, dimensions texture, are all *equally accelerated* in right l perpendicular to the plane of the horizon.

3. The apparent *inequality* of the action of vity upon different species of matter near surface of the Earth, arises entirely from the re ance which they meet with in their passage thro the air. When this resistance is removed, (a the exhausted receiver of an air-pump,) no s inequality is perceived; bodies of all kinds th descend with equal velocities; and a guinea, a feath or the smallest particle of matter, if let fall togeth are observed to reach the bottom of the rece exactly at the same instant.

4. The *weight* of a body is its gravity or tende towards the Earth's center, as referred to some fi standard. Thus, if the gravity of some given b be called an *ounce* or a *pound*, any other b having the same gravity, (or the same tendency descend, when placed under the same circu stances,) is also called an ounce or a pound;

two such bodies, taken together, *two* ounces, or *two* pounds, &c. &c.

5. Gravity, at different distances from the Earth, varies inversely as the square of the distance from its center. The weight of a given body, therefore, will vary at different heights above the Earth's surface. But the heights at which experiments are commonly made upon the weights of bodies, bears so small a proportion to the radius of the Earth, that this variation is almost imperceptible. At the height of *half a mile*, it does not amount to more than about $\frac{1}{4000}$th part of its weight at the surface[a].

1. By the INERTIA of matter, is meant "its inability to change its state of rest or motion." That matter possesses not the power of putting itself in motion, is a fact of common observation and experience. That "when once put in motion, it cannot of itself either *stop* or *alter* its motion," will be shewn from experiment, when we come to the First Law of Motion.

2. Of this inability in matter to change its state of

[a] Let r=the radius of the Earth=nearly 4000 miles; and let x=the height to which the body is carried; then its weight at the Earth's surface *(W)* : its weight at the distance x above it $::\overline{r+x}^2 : r^2 :: r^2+2rx+x^2 : r^2$:: (if x be very small when compared with r) $r^2+2rx : r^2$. Hence the weight at the Earth's surface *(W)* : the *difference* of the weights at the surface and at the distance x above it *(D)* $:: r^2+2rx : 2rx :: r+2x : 2x$; $\therefore D=\frac{2x}{r+2x}.W=$ (if $x=\frac{1}{2}$ a mile) $\frac{W}{4001}$.

of rest or motion, as existing in a given body, we have no other means of judging, than from the difficulty experienced in originating or altering motion in that body. Now in bodies of uniform texture, *i.e.* in such as are capable of being resolved into particles each having the same weight, this difficulty is found to increase with the number of such particles; from hence it appears, that in bodies of this kind, *weight* is the proper measure of inertia.

3. But this connection between weight and inertia is of the most general nature; for by experiments, performed upon the Machine invented by Mr. ATWOOD, it has been proved, that, at a given place upon the Earth's surface, the retardation of motion arising from inertia is proportional to the *weight* of a body, whatever be its figure, dimensions, or texture[a].

4. The weight or inertia of a body must evidently consist of the weight or inertia of all its parts; they may either of them, therefore, be made the measure of the *quantity of matter* which it contains. *Inertia*, however, is the fixed and proper standard to be referred to for that purpose; inasmuch as the weight of a body varies at different distances from the Earth's center, whilst its inertia is always,

(a) The definition here given of inertia differs somewhat from that which will be found in the Note, page 49, Vol. II. and from that which has been given in some other Authors. The sense in which the word is here used must be distinguished from that particular effect of inertia arising from the rotatory motion of bodies round a fixed axis, called the "*moment* of inertia," which is made the subject of calculation in the 14th Lecture.

always, and under all circumstances, the same. But at small heights above the Earth, this variation has been shewn to be very inconsiderable; for all practical purposes, therefore, the quantity of matter in a body may be considered as proportional to its *weight*.

II.

In observing the phænomena connected with the actual motion of a body, we are naturally led to consider, I. The *Space* over which it moves. II. The *Time* of its motion. III. The *Velocity* with which it moves.

1. If the motion of a body be such, that it describes equal spaces in equal successive parts of time, then it is said to move with an *uniform* velocity. When the spaces described by it in equal successive parts of time continually increase, it is said to move with an *accelerated* velocity; and with a *retarded* velocity, when those spaces continually decrease. If its motion be so regulated, that it receives equal increments of velocity in equal successive parts of time, then it is said to be *uniformly* accelerated; and uniformly *retarded*, if the body suffers equal decrements of velocity in those equal portions of time.

2. The space described by a body moving with a uniform velocity, increases in a compound proportion of the time and velocity of its motion. For let $x =$ the number of seconds for which a body moves uniformly, and let $y =$ the number of feet described in each second; then it is evident that

xy will denote the number of feet (i.e. the space) described by it in x seconds.

3. In general, if S = the space described by a body; T = the time of its motion, expressed in seconds; V = the uniform velocity with which it moves, expressed by the number of feet described in a second; and if s = the space, t = the time, v = the velocity, of any other body, expressed in the same manner; then the relation between S, T, V, and s, t, v, may be expressed by the following proportion;

$$S : s :: T \times V : t \times v.$$

4. Adopting, therefore, the Algebraic notation for variable quantities, we have $S \propto T \times V$; hence $T \propto \frac{S}{V}$, and $V \propto \frac{S}{T}$. If S be given, then $T \propto \frac{1}{V}$, and $V \propto \frac{1}{T}$.

Ex. 1. If two bodies move uniformly for 9 and 5 seconds; the former with a velocity of 17 feet, and the latter with a velocity of 10 feet, in a second; then the *spaces* described by them will be, 9×17 and 5×10, or 153 and 50 feet.

Ex. 2. The spaces described by two bodies are 540 and 320 feet; the velocities with which they move are 6 and 5 feet in a second; hence the *times* of their motion are, $\frac{540}{6}$ and $\frac{320}{5}$, or 90 and 64 seconds.

Ex. 3. Three bodies move over spaces which are to each other as 19, 17, and 12; the times of their motion are as 7, 3, and 5; consequently the *velocities* with which they move are, as $\frac{19}{7}$, $\frac{17}{3}$, and $\frac{12}{5}$, or as 285, 595, and 252 respectively.

III.

III.

The motion of a body considered with reference to the mass moved as well as its velocity, is called its *momentum*, or *quantity of motion*.

1. The momentum of a body is in the joint ratio of its quantity of matter and velocity. For since the momentum of the whole mass is the sum of the motions of all its parts, it must evidently depend upon the number of parts, and the velocity of each.

2. Let M = the momentum of a body; Q = its quantity of matter or *weight* (Art. 4, page 6), expressed in pounds; V = its velocity, expressed in feet; and let m = the momentum, q = the weight, v = the velocity, of any other body, expressed in the same manner; then the relation between M, Q, V, and m, q, v, will be expressed by the following proportion;

$$M : m :: Q \times V : q \times v.$$

3. Hence $M \propto Q \times V$; $\therefore Q \propto \frac{M}{V}$, and $V \propto \frac{M}{Q}$. If Q be given, $M \propto V$; if V be given, $M \propto Q$; and if $Q \propto \frac{1}{V}$, or $V \propto \frac{1}{Q}$, then M will be *given*; i. e. "if two bodies move with velocities which are "inversely as their quantities of matter, then their "momenta will be *equal*;" this constitutes a fundamental principle in the mechanical action of bodies.

Ex. 1. The body A weighs 7lbs. and is moving with a velocity of 9 feet in a second; B weighs 5lbs. and moves with a velocity of 11 feet in a second; then the *momentum* of A : the *momentum* of $B :: 7 \times 9 : 5 \times 11 :: 63 : 55$.

Ex. 2.

Ex. 2. The momenta of A and B are as 15 : 7, and their velocities as 10 : 3; then the *weight* of A : the *weight* of B :: $\frac{15}{10} : \frac{7}{3}$:: 9 : 14.

Ex. 3. The weights of three bodies are as 5, 3, 2; and their momenta as 1, 2, 3; then their *velocities* are as $\frac{1}{5}, \frac{2}{3}, \frac{3}{2}$, or as 6, 20, 45, respectively.

IV.

Since matter, from its inertia, is incapable of changing its state of rest or motion, it is evident that whenever such change is produced, it must be by the agency of some external cause. This cause is denominated *force* or *power*. Before we proceed to the Laws of Motion, it will be proper, therefore, to make a few observations upon the nature of FORCES in general.

1. Forces are divided into two kinds, according to the manner in which they act.

I. If a force acts *instantaneously*, and then ceases, it is called an *impulsive* force. A ball, suddenly put in motion by the hand or any instrument, along an horizontal plane, is an instance of the effect produced by an impulsive force. If it moves with a uniform velocity, the space over which it passes in a given time, and the momentum communicated to it, may be estimated according to the rules stated above.

II. When a force acts *incessantly*, it is called a *constant* or *variable* force, according as the increments or decrements of velocity caused by it, in equal successive parts of time, are *equal* or *unequal*. The force of

of gravity near the Earth's surface is an example of a *constant*[a] force; for it causes equal increments or decrements of velocity in equal portions of time, not by *impulses*, but by *incessant* action. To estimate the effects produced by it upon bodies moving freely under its influence, is the subject of Lectures II, and XII. — Gravity, considered as a branch of the general law of nature before alluded to, is an instance of a *variable* force; the *law* of whose variation is, the inverse duplicate ratio of the distance of the body from the center of the Earth. Lecture XXII. relates to the motion of bodies proceeding under the influence of a force of this kind.

2. Forces also receive different denominations, according as the *momentum* or the *velocity* generated by them is the subject of consideration. With reference to the momentum generated, they are called *moving* forces; with reference to the velocity, *accelerative* forces. This distinction between moving and accelerative forces, and the effects produced by them, forms the subject of Lectures XIII, XIV, and XV.

3. An attentive comparison of the effects produced upon a body by the single or joint action of these several kinds of forces, under all the circumstances

(a) This force cannot be considered as *absolutely* constant, except at a given distance from the Earth's center. Its *variation is* calculated in the same manner as the variation of the *weight* of a body in Note (a), page 5, and may therefore be considered as almost imperceptible for all heights above the Earth's surface, not exceeding half a mile.

cumstances in which it may be placed, has led to the establishment of three general rules or maxims; in conformity to which, every change of the state of rest or motion in a body is found, by experience, to take place. These are called LAWS of Motion, and have sometimes been considered as *axioms* in Natural Philosophy. They are, however, by no means self evident; nor are they capable of strict mathematical demonstration. The true light in which they ought to be considered, is that of *fundamental principles* in the science of Mechanics; the truth of which may be established, either by experiment, or by a just process of reasoning upon the circumstances universally connected with the GRAVITY and INERTIA of matter. They are as follow.

I. *A body continues always in its state of rest, or of uniform rectilinear motion, till, by some external force, it is made to change its state.*

II. *Motion, or change of motion, is proportional to the force impressed, and is produced in the right line in which that force acts.*

III. *When bodies act upon each other, action and re-action are equal, and in opposite directions.*

Illustration of the FIRST LAW.

1. That a body at rest will *continue* at rest, is a consequence immediately arising from the inertia of matter. That a body in motion "will continue "to proceed uniformly along the right line in "which it began to move, until it is acted upon by "some

"some external force," may be illustrated by the following experiments.

I. A spherical ball is put in motion, along horizontal planes of different degrees of smoothness, such as, an uneven pavement, a bowling-green, a sheet of ice or some polished surface. In all these cases, it *begins* to move with a velocity very nearly uniform, and in the direction of the force impressed; but its motion is *continued* in a very different manner on the different planes. On the pavement, it meets with much *collision* and *friction* from the obstacles which it encounters; in this case, therefore, its motion is very irregular, and soon ceases. In passing over the bowling-green, *friction*(a) is the principal impediment which obstructs its progress; and here we find, that it moves more regularly, and that its motion is continued for a longer time. On ice, or on any polished surface, the friction is very much diminished; and in this case it is observed to move, for a very considerable time, with still greater regularity.

II. A body is projected in a direction *oblique* to the horizon. By the action of the force of *gravity*, it is immediately made to deviate from the right line in which it would have uniformly moved from the impulse received; under the joint influence of the forces of projection and gravity, it proceeds in a curvilinear direction; it meets with considerable *resistance* in its passage through the air; and its motion is at last destroyed by *collision* at the Earth's surface.

III. A small brass wheel, of uniform thickness and texture, is put in motion round an horizontal axis. From the regularity of its construction, the different parts of the wheel just balance each other round its center, so that, in its rotatory motion,

(a) In each case the body meets with a small degree of resistance from the air; but it is very inconsiderable, when compared with the impediments arising from collision or friction.

motion, it is neither accelerated nor retarded by the force of *gravity*. The only causes, therefore, which retard its motion, are, the *friction* upon its axis, and the *resistance* of the air. If the friction be diminished by placing the axis itself upon friction wheels, its motion is continued for a much longer time than when it revolves simply upon its axis. By placing the whole apparatus in the exhausted receiver of an air-pump, the resistance of the air is removed; and then it is found to revolve with a velocity nearly uniform, for a very considerable length of time.

2. It is from a due consideration and just comparison of the results arising from a variety of plain and simple experiments of this kind, that we are enabled to establish the truth of this First Law of Motion. From the instances just adduced, it appears, that when a body is once put in motion by the agency of some force, the continuance and regularity of its motion is always increased, as we diminish the number of impediments arising from collision, friction, resistance of the air, and other retarding causes. As this is the case in every instance which comes under our notice, it seems fair to conclude, that if a body could be placed under such circumstances as to be entirely exempt from the operation of these causes, its motion would then become altogether uniform, rectilinear, and perpetual.

Illustration of the Second Law.

1. It has already been observed, that every change in the state of rest or motion in a body must be effected by the agency of some force; this Second Law asserts, that this change will in all cases be

proportional

proportional to that force, and will be produced in "the direction in which the force acts."

2. That motion, or the change of motion in a body, will be proportional to the force which produces it, i.e. "if a certain force produces a certain quantity "of motion, *twice* that force will produce *twice* the "quantity of motion, &c." is a proposition to which the mind very readily assents; it may however be proper to illustrate it by experiment. The experiments most adapted to this purpose are such as are performed by the Machine invented by Mr. ATWOOD; the principle of which is this;

I. A weight W (see Fig. page 33, Vol. II.) descends perpendicularly against the action of another weight W', attached to it by a very fine cord going over a small wheel or pulley fixed at A. In order to facilitate as much as possible the descent of W, the axis of this pulley is placed upon friction wheels; and the weights employed are so small, as to meet with little or no resistance from the air. If the weights be equal, it is evident that no motion will ensue; and by properly adjusting the relation between them, the force of descent of W may be made to bear any given ratio to the force with which it would descend freely by the action of gravity.

II. Suppose now, that, in three separate experiments, the weights W, W' should bear such a proportion to each other, that the force of descent of W should be $\frac{1}{48}$th, $\frac{1}{64}$th, and $\frac{1}{96}$th part of the force of gravity; then the spaces described in the first second, by W descending from rest, would be about 4, 3, and 2 inches, in each case respectively. The fractions $\frac{1}{48}, \frac{1}{64}, \frac{1}{96}$ are to each other as $\frac{4}{192}, \frac{3}{192}, \frac{2}{192}$, or as the numbers 4, 3, 2; in these cases, therefore, the spaces described by W are in proportion to the forces respectively impressed.

III. But this machine is so nicely contrived, that when the weight W has descended for any given time, the weight W' in its ascent is made to connect itself with another small weight, which is equal to the *difference* between the weights W, W'. The descending and ascending bodies thus becoming *equal*, the weight W will proceed on uniformly, with the velocity which it had previously acquired. In the three preceding experiments, supposing the weight W to have descended for two seconds, then the velocity acquired at the end of that time would be such as to make it move uniformly in one second over about 16, 12, and 8 inches respectively. But these numbers are to each other as 4, 3, and 2; the velocities generated in a given time are, therefore, proportional to the forces impressed upon the body.

3. From these experiments it appears, that whether we consider the motion of a body, simply as "the act of changing its place," and measure it by the space passed over in a given time; or whether we consider the *momentum* produced in it, which is measured by the velocity generated in a given time[a]; the motion or change of motion is, in each case, proportional to the force impressed.

4. The same conclusions may be drawn from comparing the spaces passed over or the velocities generated in a given time, when bodies descend freely by the action of gravity down inclined planes of different degrees of elevation; but experiments for this purpose are performed with much less facility and accuracy upon inclined planes, than

(a) The momentum of a body is measured by its quantity of matter and velocity jointly; the momentum of a *given* body, therefore, is measured by its *velocity*.

than with the machine of Mr. ATWOOD just now described.

5. With respect to the *direction* in which a body moves, it is evident, that when it is under the influence of any given force, whether it be an impulsive one, or one that acts incessantly, the body can have no tendency whatever to deviate to the one side or to the other, but must proceed along the right line in which the force acts.

6. This Second Law of Motion, as thus illustrated, applies only to the case when a body is under the *direct* operation of a single force. If a force acts *obliquely*, or if a body be acted upon at the same time by two or more forces, they must be resolved or compounded according to the principles laid down in Lect. III. before the several effects produced by them can be duly estimated in conformity with the meaning of the law.

Illustration of the THIRD LAW.

1. Before we attempt to give a particular explanation of the terms "action and re-action," as made use of in this Third Law of Motion, it will be proper to shew, by experiment, what are the effects produced by the impact of bodies upon each other.

I. Two equal bodies, whose quantities of matter or weights are respectively represented by A and B, are suspended contiguous to each other by strings of equal length. A is pulled from its perpendicular position, and made to impinge upon B at *rest*, with a velocity of a feet in a second.

Hence (Art. 1, page 9) the momentum of A before impact $= Aa$. After impact, the bodies move on together, as one

one mass, with half the velocity of A before impact. Now the *quantity of matter* in this mass $= A + B =$ (since $A = B$) $2A$; and since its *velocity* $= \frac{1}{2}a$, its *momentum* $= 2A \times \frac{1}{2}a = Aa$. The momentum of the mass *after* impact is, therefore, equal to the momentum of A *before* impact.

II. Two unequal bodies A and B are suspended in the same manner, and are both made to move in the *same* direction; A with a velocity of a feet, and B with a velocity of b feet, in a second;

$$\text{Also } A : B :: 3 : 2, \text{ or } B = \frac{2}{3}A,$$
$$\text{and } a : b :: 8 : 3$$
$$\therefore Aa : Bb :: 4 : 1$$
$$\text{and } Aa : Aa + Bb :: 4 : 5, \quad \therefore Aa + Bb = \frac{5}{4}Aa.$$

Hence the *sum* of the momenta of A and B before impact $= \frac{5}{4}Aa$.

After impact, the two bodies are observed to proceed on together with $\frac{3}{4}$ths of A's velocity before impact. Hence the momentum of the mass after impact

$$= \overline{A+B}.\frac{3}{4}a = \overline{A+\frac{2}{3}A}.\frac{3}{4}a = \frac{5}{3}A \times \frac{3}{4}a = \frac{5}{4}Aa.$$

In this case, therefore, the momentum of the mass *after* impact is equal to the *sum* of the momenta of the bodies *before* impact.

III. These two bodies are now made to impinge upon each other with the same velocities in *opposite* directions; and after impact they move on together, in A's direction, with a velocity which is found to be only $\frac{9}{20}$ths of the velocity of A before impact; $\therefore$ the *momentum* of the mass after impact $= \overline{A+B}.\frac{9a}{20} = \frac{5A}{3} \times \frac{9a}{20} = \frac{3}{4}Aa.$

Now since $Aa : Bb :: 4 : 1$, we have $Aa : Aa - Bb :: 4 : 3$, $\therefore Aa - Bb = \frac{3}{4}Aa$; hence the momentum of the mass

mass *after* impact $= Aa - Bb =$ the *difference* of the momenta of A and B before impact.

Cor. If in this experiment $A = B$, and $a = b$; or if $A : B :: \frac{1}{b} : \frac{1}{a}$; so that, in each case, $Aa = Bb$; then $Aa - Bb = 0$, and the two bodies remain at *rest* after impact[a].

2. In reviewing the results of these experiments, it appears that, in each case, the whole momentum of A and B, estimated in A's direction, is the same both *before* and *after* impact; for when A impinged upon B at *rest*, the momentum of the mass after impact was equal to the momentum of A before impact; when the bodies moved in the *same* direction it was equal to the *sum*, and when they moved in *opposite* directions it was equal to the *difference*, of the momenta of A and B before impact. From hence we infer, that in the change which takes place in the momenta of the two bodies during the act of collision, the momentum lost by A must be gained by B; for that A *loses* momentum is evident from this, that its *velocity* (which is the same with the velocity of the mass) is, in each case, *greater* before than after impact.

3. Now the moving force by which A communicates

(a) In all these cases, the bodies made use of are supposed to be perfectly hard; they are also made to move in a circular groove, to prevent their passing each other. In making the experiments, allowance must be made for the resistance of the air, and (if the bodies move through large arcs) for the irregularity arising from the acceleration or retardation of their motion by the force of gravity.

nicates momentum to *B*, is called the action of *A* and the tendency of *B* to diminish the momentum of *A* is called the *re-action* of *B*. Since, therefore according to this meaning of the words action an re-action, the effect produced by the action of *A* i equal to the effect produced by the re-action of *B* "action and re-action are said to be *equal*" durin the impact of *A* upon *B*. That these effects ar produced in "opposite directions," is evident fro the very nature of the case.

4. This law applies not only to the *impact* o bodies, but to every case in which one body act upon another. A body *A*, for instance, is sustaine by another body *B*, and both bodies remain at res if the pressure exerted by the two bodies were n *equal*, it is evident that some *motion* would ensu which is contrary to the supposition. If motio does ensue, then the case becomes, in a gre measure, analogous to that of *impact*; and the effec produced, estimated in a similar manner, are foun to observe the same law.

5. The mutual *attractions* of bodies are also sub ject to this law. Thus, if two equal magnets, con nected with two equal and similar pieces of cor be made to float upon the surface of water; as soo as they come within the sphere of attraction, the are observed to move towards each other in a rig line, with equal *velocities*, and consequently wit equal *momenta*; and as the *resistance* which eac body meets with from the fluid is evidently th

sam

same, we infer that their actions upon each other are *equal* [a].

6. It is from comparing the results arising from a variety of experiments, made for the purpose of ascertaining the exact changes which take place in the momenta of bodies during the act of collision, and for estimating the effects produced by their mutual attraction, that the truth of this Third Law of Motion has been established. These experiments are undoubtedly liable to great inaccuracy, from friction, adhesion, resistance of the air, and many other causes; but at the same time it must be observed, that they more nearly accord with the law, as the impediments arising from these causes are diminished; and that, upon the whole, they confirm all our general reasoning upon the subject.

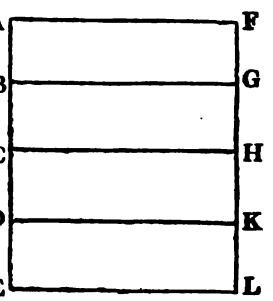
[a] If two *unequal* magnets, placed upon pieces of cork similar to each other and proportional to the respective magnets, were made to float in the same manner, they would approach each other with velocities inversely proportional to the quantities of matter moved, and consequently with *equal momenta*; but this experiment is liable to very great inaccuracy, from the different *resistance* which the bodies would meet with.

Lecture II.

ON THE ASCENT AND DESCENT OF BODIES NEAR THE EARTH'S SURFACE.

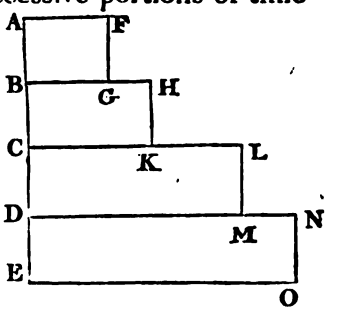
V.

1. It has already been demonstrated, that if a body moves with an uniform velocity for a given time, the space described will be in proportion to that time and velocity conjointly. If therefore one side of a right-angled parallelogram represents the *time* of a body's motion, and the other the uniform *velocity* with which it moves, the parallelogram itself will represent the *space* described in that time. Thus, let the line *AE* be divided into any number of equal parts in the points *B, C, D*, &c. and from those points draw the *equal* straight lines *AF, BG, CH*, &c. at right angles to *AE*, and complete the parallelogram *AFLE*; then if *AB, BC, CD*, &c. represent equal successive portions of *time*, and *AF, BG, CH*, &c. represent the uniform *velocity* with which a body moves, then will



will the parallelograms AG, BH, CK, &c. represent the *spaces* described in those equal portions of time, and the parallelogram $AFLE$ the *whole space* described in the time represented by AE.

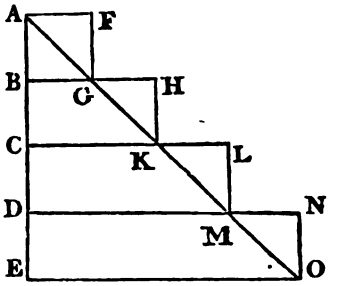
2. Suppose now that a body moves uniformly as before, during the equal successive portions of time represented by AB, BC, CD, &c. but at the end of each portion of time receives an *increase* of velocity; for instance, during the time AB let it move with a velocity represented by AF, during the time BC with a velocity represented by BH, &c. &c.; complete the parallelograms $AFGB$, $BHKC$, $CLMD$, &c. then the *space* described in the *time* AB will be represented by the parallelogram AG, in the *time* BC by the parallelogram BK, &c. &c.; and the *whole space* described in the time AE by the *irregular figure* $AFHLNOE$.



3. Let us next suppose that the body receives *equal* increments of velocity at the end of each successive portion of time, so that during the *second* interval of time it moves with *twice* the velocity, during the *third* interval with *three* times the velocity, &c. &c. then will $CK=2BG$, $DM=3BG$, &c. &c. But $AC=2AB$, $AD=3AB$, &c. &c.;

$\therefore AC$

$\therefore AC : CK :: AB : BG :: AD : DM$, &c. &c.[a]; hence if AG, GK, KM, &c. be joined, AO will be a *straight line*, and the figure $AFHLNOE$, which represents the *space* described in the *time* AE, will differ from the triangle AOE only by the sum of the triangles AFG, GHK, KLM, &c. which are all equal to each other[b].



4. Now let the intervals of time and the corresponding increments of velocity be only *half* what they were in the former instance. Bisect AB, BC, CD, &c. in b, c, d, &c. and complete the parallelograms as before, then the figure which represents the space described in the time AE will differ from the triangle AOE by the sum of the small triangles Afg, gFG, Ghk, &c. which is only *half*[c] the sum

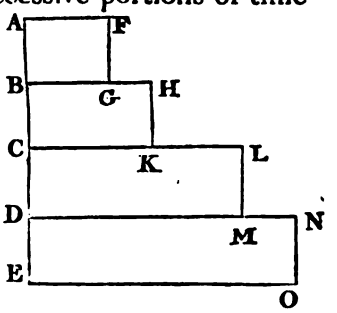
(a) For $AC : CK :: 2AB : 2BG :: AB : BG$; and $AD : DM :: 3AB : 3BG :: AB : BG :: AC : CK$, &c.; hence the figures ABG, ACK, ADM, &c. are so many *similar triangles*.

(b) For $AF=GH=KL$, &c.; and FG, HK, LM, &c. are equal to AB, BC, CD, &c.

(c) Since $AF=\frac{1}{2}AF$, and $bg=\frac{1}{2}BG$, the triangle AfG : triangle $AFG :: (Af \times fg : AF \times FG ::) \frac{1}{2}AF \times \frac{1}{2}FG : AF \times FG :: 1 : 4$; $\therefore$ the triangle Afg is only $\frac{1}{4}$ of the triangle AFG, and so of the rest. Although, therefore, the number of the triangles Afg, gFG, Ghk, &c. is *double* the number of the triangles AFG, GHK, KLM, &c. yet since $Afg=\frac{1}{4}AFG$, $gFG=\frac{1}{4}GHK$, &c. the *sum* of the former will be only *half* the sum of the latter.

will the parallelograms AG, BH, CK, &c. represent the *spaces* described in those equal portions of time, and the parallelogram $AFLE$ the *whole space* described in the time represented by AE.

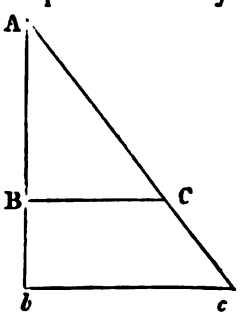
2. Suppose now that a body moves uniformly as before, during the equal successive portions of time represented by AB, BC, CD, &c. but at the end of each portion of time receives an *increase* of velocity; for instance, during the time AB let it move with a velocity represented by AF, during the time BC with a velocity represented by BH, &c. &c.; complete the parallelograms $AFGB$, $BHKC$, $CLMD$, &c. then the *space* described in the *time* AB will be represented by the parallelogram AG, in the *time* BC by the parallelogram BK, &c. &c.; and the *whole space* described in the time AE by the *irregular figure* $AFHLNOE$.



3. Let us next suppose that the body receives *equal* increments of velocity at the end of each successive portion of time, so that during the *second* interval of time it moves with *twice* the velocity, during the *third* interval with *three* times the velocity, &c. &c. then will $CK=2BG$, $DM=3BG$, &c. &c. But $AC=2AB$, $AD=3AB$, &c. &c.;

by the different parts of the triangle AOE, in such manner that if Ab, AB, Ac, &c. represent the *time* of a body's motion, and bg, BG, ck, &c. the *velocity* acquired, then the *space* described in those respective times will be represented by the triangles Abg, ABG, Ack, &c. This *geometrical* representation for the *space*, *velocity*, and *time*, furnishes us with a very easy method of investigating the relation which subsists between them.

6. For let S = the *space* described, V = the *velocity* acquired by a body falling from rest for the *time* T; s = space described, v = velocity required at any other period (t) of its fall; then, from what has already been demonstrated, if the proportion of $T : t$ be represented by the lines AB, Ab, and the proportion of $V : v$ by the lines BC, bc, drawn at right angles to them, the proportion of $S : s$ will be represented by the triangles ABC, Abc.



Now the triangle ABC : the triangle $Abc :: AB^2 : Ab^2$, or as $BC^2 : bc^2$, or as $AB \times BC : Ab \times bc$ [a].

Hence

(a) For the right-angled triangles ABC, Abc are to each other both in the duplicate ratio of their homologous sides (by Euc. 6. 19.) and in the ratio of the parallelograms of which they are respectively *halves*.

Hence $S : s :: T^2 : t^2$, or as $V^2 : v^2$, or as $T \times V : t \times v$; i. e. the *spaces* described by bodies falling from rest under the influence of gravity, are to each other as the *squares of the times* in which they are described, or as the *squares of the last-acquired velocities*, or as *the times and last-acquired velocities conjointly*. As equal increments of velocity are generated in equal times, it is further evident that the *velocity varies as the time:* the same conclusion may also be deduced from the similar triangles ABC, Abc; for $BC : bc :: AB : Ab$, i. e. $V : v :: T : t$.

7. Since the spaces described are as the *squares* of the times; if a body falls from rest for *times* which are represented by the numbers 1, 2, 3, 4, 5, &c. the *spaces* described in those times will be as the square numbers 1, 4, 9, 16, 25, &c.; and the spaces described in *equal successive portions of time* will be as the odd numbers 1, 3, 5, 7, 9, &c. as exhibited in the following table.

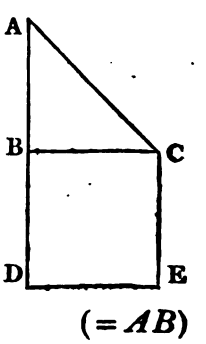
Times.	Spaces described.	Spaces described in equal successive portions of time.
1 . . .	. . 1	In *first* portion of time . . 1
2 . .	. . . 4	2d 4−1 =3
3 . . .	. . . 9	3d 9−4 =5
4 . . .	. . . 16	4th 16−9 =7
5 . . .	. . . 25	5th 25−16=9
&c. . .	. . . &c.	&c. &c. =&c.

8. As in the *descent* of a body the force of gravity generates equal increments in equal times, so in its

ascent,

ascent, equal portions of velocity will be destroyed in equal times: if, therefore, a body be projected perpendicularly upwards, with the velocity which it has acquired in falling from any height, it will rise to the point from which it fell, before it begins to descend again. The spaces described in equal successive parts of time, by a body thus ascending, reckoning from the beginning of its motion, will be the same as those stated in the foregoing table, but in an *inverted* order: thus, if the time be divided into four equal parts, then the spaces described in the *descent* of the body during these equal times are as the numbers 1, 3, 5, 7, but in its *ascent* they will be as 7, 5, 3, 1; i. e. the space described in the *first* portion of time, in its *ascent*, will be the same as that described in the *last* portion of time, in its *descent*, &c. &c. till the body arrives at its highest point.

9. The space which a body describes from rest in any time, by the action of gravity, is *half* that which it would describe in the same time with the last-acquired velocity continued uniformly. Let the triangle ABC represent the *space* described by gravity in the *time* AB, and BC the last-acquired *velocity*; produce AB to D, making BD equal to AB, and complete the parallelogram $BCDE$; then, if a body moves for the time BD



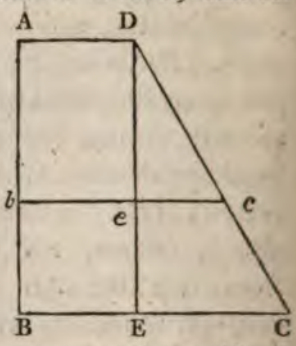
($=AB$) with the uniform velocity represented by BC, the space described in that time will be represented by the parallelogram $BCDE$; but the triangle ABC is *half* the parallelogram $BCDE$; hence the space described with the continually increasing velocity during the time AB is *half* that which would be described in the same time (BD) with the velocity (BC) continued uniformly. From this also it follows, that if a body, after it has fallen from rest through any space, should then proceed on *uniformly* with the last-acquired velocity, it will describe *twice* the space in the same time as that in which it has fallen to acquire that velocity[a].

10. Instead of the body's beginning to fall from *rest*, suppose now that it was projected downwards with a given velocity, and that it was required to find the space described by it in some given time. Let AD represent the given *velocity* of projection, and

AB

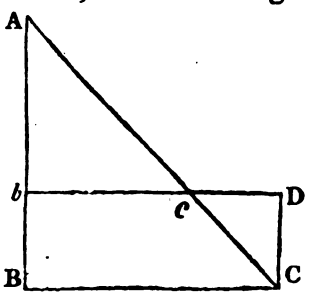
(a) Since the space described by a body falling from rest is *half* that which it would describe in the same time with its greatest velocity continued uniformly, and since a body projected perpendicularly upwards rises to the same height as that from which it must fall to acquire the velocity of projection, the space described in any time by a body so projected is *half* that which it would describe in the same time with its first velocity continued uniformly.

AB the given *time*, and complete the right-angled parallelogram *ADEB*; produce *BE* to *C*, and let *EC* represent the velocity generated by gravity in the time *AB* or *DE*, and join *DC*. Draw any line *bc* at right angles to *AB*, cutting *DE* in *e*; then it is evident that *bc* will represent the whole velocity of the body at the end of the time *Ab* or *De*; for *be* (= *AD*) represents the uniform velocity with which it began to move, and *ec* represents the velocity generated by gravity in the time *Ab* or *De*. The parallelogram *ADeb*, moreover, represents the space which would be described with the uniform velocity (*AD*) in the time *Ab*, and the triangle *Dec* represents the space described by *gravity* in the same time. Hence the *whole space* described in the time *Ab* by the body thus projected, will be represented by the figure *ADcb*; and for the same reason the space described in the time *AB* will be represented by the figure *ADCB*, which is made up of the parallelogram *ADEB* and the triangle *DEC*. The space described therefore in any time by a body projected downwards with a given velocity, is equal to *the space which would be described with that velocity continued uniformly for that time*, together with *the space through which a body would fall from rest by the action of gravity in the same time.*



11. We

11. We have already observed, that if a body be projected perpendicularly upwards with any velocity, it will rise to the same height from which it must fall to acquire that velocity. Let BC represent the given velocity of projection, and AB the time in which it must fall from rest to acquire that velocity; draw BC at right angles to AB, and join AC, then the triangle ABC will represent the space through which the body must ascend to lose all its velocity. In AB take any point b, and complete the parallelogram $BCDb$; then will bc represent the velocity of the body at the end of the time Bb of its ascent, and cD $(=bD-bc=BC-bc)$ will represent the velocity destroyed by gravity in the same time. But the velocity *destroyed* by gravity in any time is equal to the velocity *generated* by gravity in the same time; hence the triangle CDc will represent the space through which a body would fall from rest in the time CD or Bb. Now the figure $BCcb$ represents the space through which the body would ascend in the time Bb, and the parallelogram $BCDb$ represents the space through which a body would move in the time Bb with the velocity BC continued uniformly; but the figure $BCcb$ is equal to the *difference* between the parallelogram $BCDb$ and the triangle CDc; hence the space described by a body ascending for a given time is equal to the *difference* between the *space*



which

which would be described by the body moving uniformly for that time with the velocity of projection, and the *space through which a body would fall from rest by the action of gravity in the same time.*

Having thus investigated some of the general principles which regulate the motion of bodies descending or ascending perpendicularly near the surface of the earth, we now proceed to the practical application of them.

VI.

On the method of finding the actual space, velocity, and time of bodies descending or ascending perpendicularly near the Earth's surface.

In estimating the actual motion of bodies descending or ascending by the force of gravity, it is necessary to have recourse to some *fixed standard* of measurement of *space* and *velocity*. Now it has been ascertained, by the most accurate experiments, that a body falling freely [a] from *rest* describes a space

(a) All bodies descending or ascending near the surface of the earth meet with more or less resistance from the air; so that, strictly speaking, a body can never be said to descend *freely* but in the exhausted receiver of an air-pump. It is in vacuo that a body describes $16\frac{1}{12}$ feet in a second; the conclusions, therefore, deduced in this section, will approximate to the truth only in those cases where the resistance of the air bears little or no proportion to the weight of the body.

space equal to $16\frac{1}{12}$ feet in the *first second* of its fall; and by Art. 9. Sect. V. a body so falling would acquire a velocity which, if continued *uniformly*, would carry it over $32\frac{1}{6}$ feet (i. e. *twice the space*) in the same time.

1. If, therefore, $m = 16\frac{1}{12}$ feet, m will express the space fallen through from rest in 1″, and $2m$ will express the *velocity per second* acquired in that time. Let S = space described by the body in any other *time* T, and V the velocity acquired; then, since (by Art. 6. Sect. V.) the spaces are as the *squares of the times*, we have,

$S : m :: T^2 : 1^2$, $\therefore S = m T^2$, and $T^2 = \frac{S}{m}$ or $T = \sqrt{\frac{S}{m}}$.

The spaces are likewise as *the squares of the velocities*, hence $S : m :: V^2 : \overline{2m}|^2$ $(4m^2)$, $\therefore S = \frac{mV^2}{4m^2} = \frac{V^2}{4m}$, and $V^2 = 4mS$ or $V = 2\sqrt{mS}$. Again, since *the time varies as the velocity*, we have $T : 1 :: V : 2m$, $\therefore V = 2m T$, and $T = \frac{V}{2m}$. From which it appears, that of the three quantities S, T, V, any *one* being given, the *other two* may be found; for

If T be given, then $S = mT^2$, and $V = 2mT$.

V $S = \frac{V^2}{4m}$, and $T = \frac{V}{2m}$.

S $T = \sqrt{\frac{S}{m}}$, and $V = 2\sqrt{mS}$.

Ex. 1.

A body has been falling for 6 seconds; What *space* has it fallen through in that time, and what is the *velocity* which it has acquired?

Here $T=6$, $\therefore\ S=mT^2=16\frac{1}{12}\times 36=579$ feet, for the space fallen through;

and $V=2mT=32\frac{1}{6}\times 6=193$ feet, for the velocity acquired per second.

Ex. 2.

How far must a body fall to acquire a velocity of 50 feet in a second; and how long will it be falling?

Here $V=50$, $\therefore S=\frac{V^2}{4m}=\frac{2500}{4\times 16\frac{1}{12}}=\frac{7500}{193}=38.86$ feet.

$$T=\frac{V}{2m}=\frac{50}{32\frac{1}{6}}=\frac{300}{193}=1.55 \text{ seconds.}$$

Ex. 3.

A body fell from the top of a tower which was 150 feet high; How long was it in falling, and what velocity had it acquired when it got to the bottom?

Here $S=150$, $\therefore T=\frac{\sqrt{S}}{m}=\frac{\sqrt{150}}{16\frac{1}{12}}=3.05$ seconds,

and $V=2\sqrt{mS}=2\sqrt{16\frac{1}{12}\times 150}=98.22$ feet per second.

Ex. 4.

A body was projected perpendicularly upwards with

with a velocity of 100 feet in a second; How far would it ascend before it began to return?

By Art. 8. Sect. V. the height to which the body would ascend, is equal to that through which a body must fall from rest to acquire the velocity of projection; Here therefore $V=100$, & $S=\frac{V^2}{4m}=\frac{10000}{4\times 16\frac{1}{12}}$ $=\frac{30000}{193}=155.4$ feet.

Ex. 5.

A body was observed to fall for 3 seconds, and afterwards to move uniformly for 2 seconds along the horizon with the velocity which it had acquired by its fall; What was the whole space described in its perpendicular and horizontal motion?

The space described in its fall $=mT^2=16\frac{1}{12}\times 9$ $=144\frac{3}{4}$ feet. The velocity acquired $=2mT=32\frac{1}{6}\times 3$ $=96\frac{1}{2}$ feet per second; and as it moved along the horizon for 2 seconds with this velocity, it must in that time have described 193 feet; hence the whole space described from the beginning of its fall to the end of its horizontal motion is $144\frac{3}{4}+193$, or $337\frac{3}{4}$ feet.

2. Since the spaces described in equal successive parts of time (by Art. 7. Sect. V.) are as the odd numbers 1, 3, 5, 7, 9, &c., and since the space described

described by a body falling from rest is in the *first* second m feet, the space described in *successive* seconds will be m, $3m$, $5m$, $7m$, $9m$, &c. feet.

Ex. 1.

A body has been falling for 5 seconds; Compare the spaces described in the *third* and *fifth* seconds of its fall.

The space described in the *third* second $= 5m = 5 \times 16\frac{1}{12} = 80\frac{5}{12}$ feet; the space described in the *fifth* second $= 9m = 9 \times 16\frac{1}{12} = 144\frac{3}{4}$ feet.

Ex. 2.

A body had fallen through 579 feet; What was the space described by it in the *last* second?

We must first find the number of seconds for which the body has been falling; Now $T = \sqrt{\frac{S}{m}} = \sqrt{\frac{579}{16\frac{1}{12}}} = \sqrt{\frac{6948}{193}} = \sqrt{36} = 6$ seconds; the *sixth* term in the progression is $11m$, $\therefore$ the space described in the *last* second $= 11m = 11 \times 16\frac{1}{12} = 176\frac{11}{12}$ feet.

3. The method adopted in the last example for finding the space described by a body in the *last* second of its fall, is only applicable when the time consists of a *determinate* number of seconds; but it is not difficult to investigate a *general* expression for

for the space described in the *last* n seconds, whatever be the value of T. For the *space* described in T seconds $= mT^2$; and the space described in $T-n$ seconds $= m \times \overline{T-n}|^2 = mT^2 - 2mnT + mn^2$; hence the space described in the last n seconds $= mT^2 - \overline{mT^2 - 2mnT + mn^2}$ (a) $= 2mnT - mn^2 = m \cdot \overline{2nT - n^2}$; if $n = 1$, then the space described in the *last* second $= m \cdot \overline{2T-1}$.

If it were required to find the space described in the second *immediately previous* to the last n seconds, we have,

Space described in last n seconds $= m \cdot \overline{2Tn - n^2}$ (*A*)

Ditto in last $\overline{n+1}$ seconds $= m \cdot \overline{2T \cdot \overline{n+1} - \overline{n+1}|^2}$ (*B*).

Subtract (*A*) from (*B*), then the space described in the second immediately previous to the last n seconds $= m \cdot \overline{2T - 2n - 1}$.

Ex. 1.

A body has fallen for $10\frac{1}{2}$ seconds; What was the space described in the last second of its fall?

Here $T = 10\frac{1}{2}$, $\therefore m \cdot \overline{2T-1} = 16\frac{1}{12} \times 20 = 321\frac{2}{3}$ feet.

Ex. 2.

(a) For the space described in the last n seconds is equal to the difference between the *whole* space and the space described in $T-n$ seconds.

Ex. 2.

What was the space described in the last 2 seconds by a body which had fallen from the top of a tower 300 feet high?

Here $T = \left(\sqrt{\frac{S}{m}} =\right) \sqrt{\frac{300}{16\frac{1}{12}}} = 4 . 3$ seconds, and $n = 2$; hence $m . \overline{2nT - n^2} = 16\frac{1}{12} \times \overline{17 . 2 - 4} = 16\frac{1}{12} \times 13 . 2 = 212 . 29$ feet.

Ex. 3.

A body has been falling for $9\frac{1}{2}$ seconds; What was the space described in the last second but 3 of its fall?

Here $T = 9\frac{1}{2}$, $n = 3$ } $\therefore$ The space described in the second *immediately previous* to the last 3 seconds $= m . \overline{2T - 2n - 1} = 16\frac{1}{12} \times 12 = 193$ feet.

4. We must next find the space described in a given time by a body *projected upwards or downwards with a given velocity*. Let V = the given velocity with which a body is projected *downwards*, and T = the time of its motion; then the space described in the time T with the uniform velocity V will be equal to $T \times V$, and the space through which a body would fall by gravity in the same time is mT^2; hence, from what was shewn in Art. 10. Sect. V. the space described in the time T by a body projected *downwards* with the velocity V is

is equal to $T \times V + mT^2$; and applying the same process of reasoning to Art. 11. of the same section, the space through which a body would ascend in the time T, if projected *upwards* with a given velocity V, will be equal to $T \times V - mT^2$.

Ex. 1.

A body is projected *downwards* with a velocity of 30 feet in a second; How far will it fall in 4 seconds?

Here $V = 30$, $T = 4$ $\Big\}$ $\therefore T \times V + mT^2 = 4 \times 30 + 16\frac{1}{12} \times 16 = 120 + 257\frac{1}{3} = 377\frac{1}{3}$ feet.

Ex. 2.

A body is projected *upwards* with a velocity of 120 feet in a second; How far will it rise in 3 seconds?

Here $V = 120$, $T = 3$ $\Big\}$ $\therefore T \times V - mT^2 = 3 \times 120 - 16\frac{1}{12} \times 9 = 360 - 144\frac{3}{4} = 215\frac{1}{4}$ feet.

5. We shall conclude this Lecture with a few examples of a miscellaneous kind.

Ex. 1.

With what velocity must a body be projected downwards from the height (a), that it may describe it in T seconds?

Let x = the velocity required; then the space described

described by a body projected downwards with velocity (x) in the time (T) is $Tx+mT^2$; hence $Tx+mT^2=a$, $\therefore x=\frac{a-mT^2}{T}$. For instance, let $a=150$, and $T=2$, then the velocity with which a body must be projected downwards from the top of a tower whose height is 150 feet, so that it may arrive at the bottom in *two* seconds, is $\left(\frac{a-mT^2}{T}\right)$ $\frac{150-16\frac{1}{12}\times 4}{2}=\frac{150-64\frac{1}{3}}{2}=42\frac{5}{6}$ feet in a second.

Ex. 2.

The space described by a heavy body in the 4th second of its fall, was to the space described in the last second, except 4, as 1 to 3; What was the whole space described by the body?

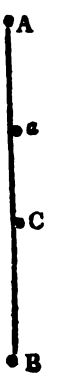
The space described in the 4th second $=7m$; the space described in the last second but 4 $(=m\,.\,\overline{2T-2n-1})=m\,.\,\overline{2T-9}$, where $T=$ the *whole time* of falling; hence from the question $7m : m\,.\,\overline{2T-9} :: 1 : 3$, $\therefore 2T-9=21$, or $T=15$; the whole space described therefore $(=mT^2)=16\frac{1}{12}$ $\times 225=3618\frac{3}{4}$ feet.

Ex. 3.

Suppose at the same instant that a body begins to fall from rest from the point (a), another body is

is projected upwards with a velocity which would carry it to A; It is required to find the point where they would meet.

Let C be the point where the bodies would meet; and let $AB=a$, $aB=b$, $aC=x$; then will $Aa=a-b$, $AC=a-b+x$. Now the time of *descending* through $aC=\sqrt{\frac{x}{m}}$; and the time of *ascending* through BC (= time down AB − time down AC)[a] $=\sqrt{\frac{a}{m}}-\sqrt{\frac{a-b+x}{m}}$; but the time *down* aC must be equal to the time *up* BC; hence we have,



$$\sqrt{\frac{x}{m}}=\sqrt{\frac{a}{m}}-\sqrt{\frac{a-b+x}{m}},$$

$$\text{or } \sqrt{x}=\sqrt{a}-\sqrt{a-b+x};$$

$$\therefore \sqrt{a-b+x}=\sqrt{a}-\sqrt{x},$$

$$\text{and } a-b+x=a+x-2\sqrt{ax};$$

$$\therefore 2\sqrt{ax}=b, \text{ or } 4ax=b^2, \text{ and } x=\frac{b^2}{4a}.$$

Ex. 4.

Suppose a body to have fallen from A to B, (Fig. in page 42.) when another body begins to fall from rest at (a); how far will the *latter* body fall before it is overtaken by the *former*?

Let

(a) For the time of *ascending* through BC = time of *descending* through it after the body has fallen from A.

Let C be the point where one body overtakes the other, and let $AB=a$, $Ba=b$, $aC=x$; then $AC=a+b+x$. Now time down $aC=\sqrt{\frac{x}{m}}$, and time down $BC=$ time down AC $-$ time down $AB=\sqrt{\frac{a+b+x}{m}}-\sqrt{\frac{a}{m}}$; but at the moment when the lower body is overtaken,



Time down $aC=$ time down BC,

$$\text{or } \sqrt{\frac{x}{m}}=\sqrt{\frac{a+b+x}{m}}-\sqrt{\frac{a}{m}};$$

$$\therefore \sqrt{x}+\sqrt{a}=\sqrt{a+b+x},$$

$$\text{and } x+a+2\sqrt{ax}=a+b+x,$$

$$\text{or } 2\sqrt{ax}=b, \ \therefore x=\frac{b^2}{4a}.$$

VII.

Questions for Practice.

1. When a body has fallen from rest for 7 seconds; what *space* has it passed through, and what *velocity* has it acquired per second?

Answer .. $S=788\frac{1}{12}$ feet; $V=225\frac{1}{6}$ feet per second.

2. How far must a body fall to acquire a *velocity* of 90 feet in a second; and what will be the *time* of it falling?

Ans. $S=125\,.\,9$ feet; and $T=2\,.\,79$ seconds.

3. A

3. A body fell into a well which was 250 feet deep; *How long* was it in its fall, and what *velocity* had it acquired at the end of it?

Ans. $T=3\,.\,9$ seconds; $V=126\,.\,8$ feet in a second.

4. A body is projected perpendicularly upwards with a velocity of 40 feet in a second; How far will it ascend before it begins to return?

Ans. 24.8 feet.

5. A body having fallen for $3\frac{1}{2}$ seconds, was afterwards observed to move along the horizon (with the velocity which it had acquired in its descent) for $2\frac{1}{2}$ seconds; What was the whole space described by the body from the beginning of its fall?

Ans. $478\frac{1}{2}$ feet, very nearly.

6. What was the space described in the *last* second by a body which had fallen for 7 seconds?

Ans. $209\frac{1}{12}$ feet.

7. A body has fallen from the top of a tower 330 feet high; What was the space described by it in the last 3 seconds? Ans. 291.3 feet.

8. Suppose a body to be projected *downwards* with a velocity of 17 feet in a second; How far will it fall in 5 seconds? Ans. $487\frac{1}{12}$ feet.

3. A

9. A body is projected *upwards* with a velocity of 55 feet in a second; How far will it rise in 2 seconds? Ans. $45\frac{2}{3}$ feet.

10. With what velocity must a stone be projected into a well 350 feet deep, that it may arrive at the bottom in 4 seconds?

Ans. $V = 23\frac{1}{6}$ feet in a second.

11. Upon a steeple 150 feet high, is a spire of 40 feet; at the same instant that a stone was let fall from the *top* of the steeple, another was projected perpendicularly upwards from the *bottom* of it, with a velocity sufficient to carry it to the top of the spire; At what point will these stones meet?

Ans. 29 . 6. feet from *the top of the steeple.*

12. Upon the top of a tower 200 feet high, is placed a flag-staff of 27 feet; a bullet is let fall from the *top* of this flag-taff; and at the instant of its passing the *bottom* of it, a stone is let fall from a window 45 feet from the top of the tower; At what distance from the bottom of the tower will the bullet overtake the stone?

Ans. 136 . 25 feet.

Lecture III.

ON THE

COMPOSITION AND RESOLUTION OF MOTION.

In the two preceding Lectures we have only considered the motion produced in bodies by the action of a *single* force. We now proceed to shew the manner in which a body would move when acted upon at the same time by *several* forces.

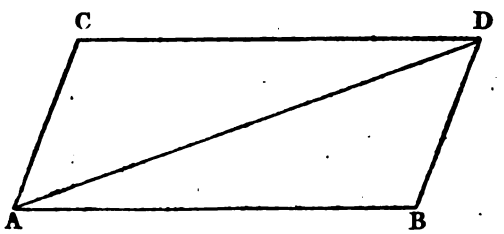
VIII.

On the Composition of Motion.

When a body is acted upon at the same time by two or more forces whose directions are not in the same straight line, it is evident that it will deviate from the course in which it would have moved by the single action of either of those forces, and will proceed in some intermediate direction. Let us first consider the case of a body acted upon by *two* forces.

1. Suppose a body placed at A to be acted upon by two forces, *one* of which would cause it to move uniformly over the line AB, and the *other* over the line

line AC in the same given time, then complete the parallelogram $ACDB$; and if both forces act at the same instant upon the body, it will, by their *joint* action, move uniformly over the diagonal AD in the same time that it would have described either of the sides AB or AC by the forces acting *separately*. For if we attentively consider the manner in which these

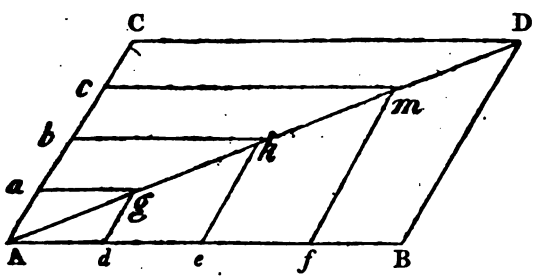


forces act, it is evident that the force which acts in the direction AB can have no tendency whatever to prevent the access of the body towards the line CD which is parallel to AB. When both forces act together, therefore, it will, by the action of the force in direction AC, arrive at the line CD (but in a *different point* of that line) in the same time as if the force in direction AB had not acted; for the same reason, the force in direction AC will have no tendency to prevent the access of the body towards BD which is parallel to AC; it will arrive therefore at the line BD in the same time as if the force in the direction AC had not acted. Hence the body will arrive at the lines CD and BD at the same instant of time, and consequently will be found at their common intersection D; and as the body, after

it

it leaves the point *A*, is acted upon by no external force, it must, by the *first law of motion*, have described the diagonal *AD* with an uniform motion.

2. As this motion of a body in the *diagonal* of a parallelogram, *by the joint action of two forces which (acting separately) would have caused it to describe the two sides*, is a fundamental theorem with respect to the *composition of motion*, it may be worth while to consider it in another point of view. Let the lines *AC*, *AB* be divided into the same number of small equal parts, *Aa*, *ab*, *bc*, &c.; *Ad*, *de*, *ef*, &c.



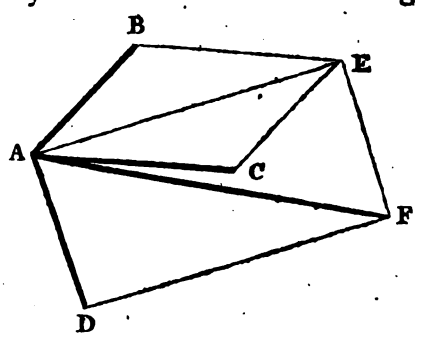
which will be to each other as the whole lines *AC*, *AB*, i. e. *Aa* : *Ad* :: *AC* : *AB*; *ab* : *de* :: *AC* : *AB*; &c. and consequently *Aa* + *ab* (*Ab*) : *Ad* + *de* (*Ae*) :: *AC* : *AB*; &c, &c.; if therefore the parallelograms *Adga*, *Aekb*, &c. be completed, then (by Euc. 6. 26.) the points *g*, *k*, *m*, &c. will all fall in the diagonal *AD*. Now since *AC*, *AB* are described uniformly in the same time, the proportional parts *Aa*, *Ad*; *Ab*, *Ae*, &c. will be described uniformly in the same time. From what has already been demon-

demonstrated therefore, at the end of those different parts of time the body will be brought to the points g, k, m, &c. by the united action of the forces which would have separately made it move over Aa, Ad; Ab, Ae, &c. Let the number of parts into which AC, AB are divided be *indefinite*, then the number of points g, k, m, &c. will be indefinite, and the lines Ag, gk, km, &c. will be indefinitely small; the body therefore will *begin* to move in the line Ag, and, being found at the end of each successive instant of time in the line AD, it must have moved over that line with the uniform velocity with which it set off.

3. From hence it follows, that if a body be acted upon by two forces, one of which would cause it to move uniformly over the side AC and the other over the side CD, of the *triangle* ACD, then, by the joint action of those forces, it would be made to describe the third side AD in the same time that it would have described either of the sides AC, CD by the forces acting separately. For if the parallelogram $ACDB$ be completed, then, since AB is equal and parallel to CD, a force acting in the direction AB would make a body describe AB in the same time as that in which it would describe CD; but by what has already been proved, if two forces act upon a body by one of which it would be made to describe AC, and by the other AB, in the same time, it would by the joint action

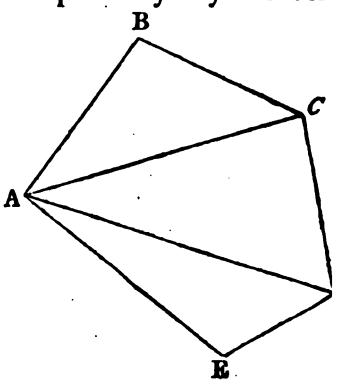
action of these forces be made to describe the diagonal AD, which is the *third side* of the triangle ACD.

4. Let us next suppose a body placed at A, and acted upon by *three* forces, by *one* of which it would be made to describe AB, by *another* AC, and by the *third* AD, uniformly in the same time; complete the parallelogram $ABEC$, and join AE; complete also the parallelogram $AEFD$, and join AF; then AF will be the line over which the body will move uniformly by the joint action of those forces, in the same time in which it would have described AB, AC or AD by either of the forces acting separately. For by Art. 1. a body acted upon by two forces in directions AB, AC would be made to describe the diagonal AE; a body placed at A therefore, and acted upon by *three* forces in directions AB, AC, AD, is under the same circumstances as if it was acted upon by *two* forces, one of which would make it describe AE, and the other AD, in the same time; but the line over which a body would move uniformly by the action of two forces in the directions AE, AD is the diagonal AF; AF therefore is the line over which it would move uniformly by the joint



action of the three forces in the directions A
AC, AD.

5. Since BE is equal and parallel to A
(Fig. in page 49.) and EF equal and parallel
AD, it follows, (for the same reason as in Art. 3
that, "if a body be acted upon by three force
"each of which acting separately would make it d
"scribe, in *succession*, the three sides AB, BE, EF
"the figure $ABEF$, taken in the order of the lette
"A, B, E, F; it would by the *joint action* of tho
"forces be made to describe the fourth side AF
"the same time that it would have described eith
"of those sides when the forces act separately." (
in general, "if a body be impelled by any number
"forces which, acting
"*separately*, would, in
"a given time, make it
"describe each of the
"sides AB, BC, CD,
"DE of the *polygon*
"$ABCDE$; when all
"those forces act *at the*
"*same instant*, it will
"be made to describe



"the remaining side AE in the same given time.

6. By the second law of motion *the space*
scribed is proportional to the force impressed; in
these cases, therefore, the *spaces* respectively
scribed by the body will represent the *quantity a*
direction of the forces by which it is impelle

Th

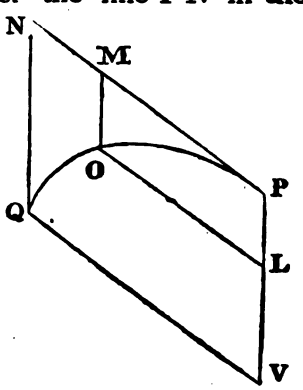
Thus (see Fig. in Art. 1.) if the *quantity and direction of two forces* be represented by the two sides AB, AC of the parallelogram $ACDB$, the diagonal AD will represent the quantity and direction of a force *equivalent to them both;* or if the *two sides* AC, CD of the triangle ACD represent the quantity and direction of two forces acting at the same time upon a body, the *third side* AD will represent a force equivalent to them both. With respect also to the forces by which a body is made to describe the sides AB, BC, CD, DE, of the polygon $ABCDE$ (Fig. in Art. 5.); if AB, BC, CD, DE represent the quantity and direction of several forces acting at the same instant upon a body, the *remaining side* AE will represent the quantity and direction of a force *equivalent to them all.*

7. Since the lines which represent the proportion of the *forces* in these different figures are described in the *same time,* and since the velocity of a body is proportional to the space described in a given time, these lines will also represent the proportion of the *velocities* with which they are respectively described. Thus (Fig. Art. 1.) *the velocity with which the diagonal* AD *is described : velocity with which either of the sides* AC *or* AB *is described* :: $AD : AC$ *or* AB; and in the case of the polygon in Art. 5, the *velocity with which the side* AE *is described : velocity with which either of the sides* AB, BC, CD, *or* DE *is described* :: $AE : AB$, BC, CD, *or* DE.

8. Hitherto

8. Hitherto the forces have been supposed to be such as by their *separate* action would produce *uniform* velocities; in which case, a body, by their *joint* action, will be made to describe a straight line with an uniform velocity. But if two forces act upon a body, by one of which it would be made to describe a straight line with an *uniform* velocity, and by the other with a *variable* velocity, then the body, by the united action of those forces, will neither describe a *straight* line, nor will it move with an *uniform velocity*; but will describe with a variable velocity some *curve* line, the form of which must be determined from the particular nature of the two forces which act separately upon the body. Let us take the case of a body projected obliquely at the earth's surface, on supposition that it meets with no resistance in its passage through the air.

9. Conceive a body to be projected from the point *P*, in the direction *PN*, with such a velocity as would carry it uniformly over the line *PN* in the same time that it would descend by the force of gravity through the space *PV*. Complete the parallelogram *PNQV*; then for the same reason as in Art. 1, the body at the end of that given time would be found in the point *Q*;



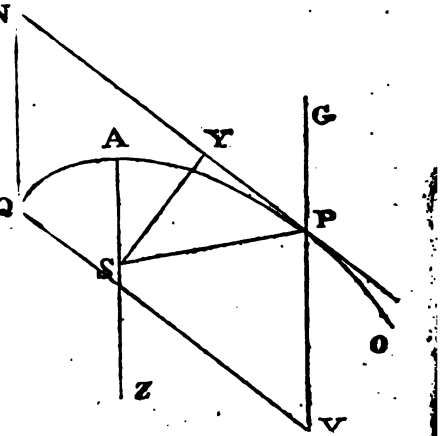
having

having described *not the diagonal* PQ, but some *curve line* POQ. In PN take any point M, and let T, t represent the times of describing PN, PM respectively; make PL equal to the space through which a body would fall by gravity in the time (t), and complete the parallelogram $PMOL$; then O will be the place of the body at the end of the time (t); and in the same manner the other points of the curve POQ might be determined. Now since PN is described with an uniform velocity, $PN : PM :: T : t$, $\therefore PN^2 : PM^2 :: T^2 : t^2$; and since NQ, MO are the spaces fallen through by gravity in the time T, t, (by Art. 6. Sect. V.) $NQ : MO :: T^2 : t^2$; hence $NQ : MO :: PN^2 : PM^2$, or the curve is of such a nature, that $MO \propto PM^2$, which (by Prop. 9. Lect. 2. Con. Sect.) is the property of the *Parabola*. The curve POQ, therefore, is a parabola whose *diameter* is PV, *ordinate* QV, and whose *parameter* to the point P is $\frac{QV^2}{PV}$ (a) or (since $PN = QV$, being opposite sides of a parallelogram) $\frac{PN^2}{PV}$.

10. If it be required to find the *dimensions* of the parabola thus described, produce VP upwards towards G, and make the angle NPS equal to the angle

(a) By Cor. to Prop. 8 of *Parabola*, parameter $\times PV = QV^2$, $\therefore$ parameter $= \frac{QV^2}{PV}$.

angle NPG[b]; take PS equal to $\frac{PN^2}{4PV}$[c], and S will be the *focus*. Through S draw AZ parallel to PV, then AZ will be the *axis* and A the *vertex* of the parabola. Let fall the perpendicular SY upon the tangent PN, then $SP : SY ::$ rad. : sin. $\angle SPY$, and if rad. $= 1$, $SY = SP \times$ sin. $\angle SPY$. By Prop. 12. of Parabola, $SP : SY :: SY : SA$, $\therefore SA = \frac{SY^2}{SP} = \frac{1}{4}$ the *latus-rectum*, from which the latus-rectum itself is known, and consequently the *dimensions of the parabola*.



11. It only now remains to find the *velocity* of the projectile. Suppose, therefore, that the body is in the act of describing the parabola OPA, and that it

(b) Since PN is parallel to the ordinate QV, it is a *tangent* to the parabola in the point P, and (by Prop. 2. of the Parabola) the tangent *bisects* the angle GPS.

(c) By Prop. 6. of Parabola, $4SP \times PV = QV^2$, $\therefore SP = \frac{QV^2}{4PV} = \frac{PN^2}{4PV}$.

it is arrived at the point P. Draw PV perpendicular to the horizon, and PN a tangent to the parabola in P. Let PV be the space through which a body must fall from rest by the force of gravity to acquire the velocity which it has in the curve at P, and take PN equal to the space which it would describe in the same time *with that velocity continued uniformly*; then (by Art. 9. Sect. V.) $PN = 2PV$. Complete the parallelogram $PNQV$, and since NQ $(=PV)$ is the space through which the body must fall by the force of gravity in the same time that it would move uniformly over PN, Q is a point in the parabola. Now, by the property of the parabola, $4SP \times PV = QV^2 = PN^2 =$ (since $PN = 2PV$) $4PV^2$, $\therefore$ (dividing by $4PV$) $SP = PV$; hence, since PV is equal to SP, the velocity in any point P is such as a body would acquire in falling from rest through (SP) i. e. through *one fourth part of the parameter* to the point P. The velocity at the *vertex* A is therefore such as a body would acquire in falling through $\frac{1}{4}$*th the latus-rectum.*

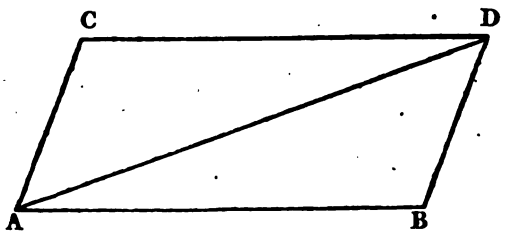
The ~~subject~~ motion of projectiles will ~~be resumed in~~ form the object of the Sixth Lecture. I shall now endeavour to illustrate this theory of the *Composition of Motion* by a few plain examples.

Ex. 1.

A body is acted upon at the same time by two forces which are to one another as $a : b$, and their directions are inclined to each other in the given angle

angle A; what is the magnitude of the *compound* force?

Let $AC : AB$ represent the ratio of $a : b$, and let BAC be equal to the given angle A. Complete the parallelogram $ABDC$, then AD will represent

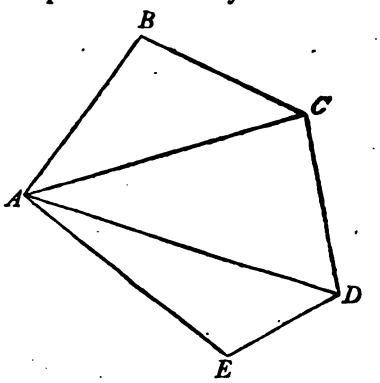


the *compound* force. Since $CD = AB$, $AC : CD :: a : b$; and since CD is parallel to AB, the angles BAC, $ACD = 180°$, $\therefore ACD = 180° - \angle A$; hence the problem is reduced to the finding *trigonometrically* the *third* side AD of the triangle ACD, in which are given the *two sides* AC, CD, and the *included angle* ACD.

Let $AC : AB :: 2 : 3$, and $\angle A = 60°$, $\therefore \angle ACD = 120°$; then (Trig. Sect. XXVIII. Case 3.) $5 : 1 ::$ tan. $30° :$ tan. $\frac{1}{2}\overline{CAD - \frac{1}{2}CDA} = \frac{1}{5}$ tan. $30°$; $\therefore$ log. tan. $\frac{1}{2}\overline{CAD - \frac{1}{2}CDA}$ = log. tan. $30°$ − log. 5 = 9.0624694 = log. tan. $6° 34'$; hence $\angle CAD = 36° 34'$, and $\angle CDA = 23° 26'$, and as sin. $29° 26'$ (CDA) : sin. ACD $(120°) :: AC$ $(2) : AD = 4.35$, i. e. if two forces which are to each other as $3 : 2$ act upon a body at an $\angle$ of $60°$, the *compound* force will be proportional to 4.35.

This

This method likewise applies to the finding the magnitude of the force compounded of any number of forces whose quantities and directions are represented by the sides of the given polygon $ABCDE$. For since AB, BC and the $\angle ABC$ are given, AC and the $\angle BCA$ may be found; but $\angle ACD = \angle BCD - \angle BCA$,

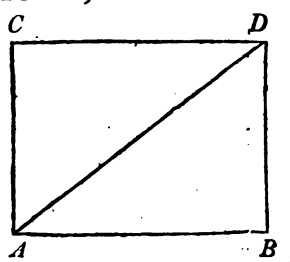


$\therefore$ AC, CD and $\angle ACD$ are known, from which AD and $\angle ADE$ may be determined; and in the triangle ADE we have AD, DE, and the $\angle ADE$, $\therefore$ AE is known.

Ex. 2.

Two forces at right angles to each other act upon a body; by one of them it would be made to move over 4 yards, and by the other 3 yards, in the same time; What space would it move over in that time by the joint action of these forces?

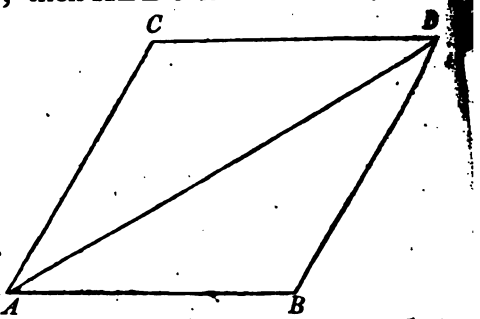
Here let AB or $CD = 4$, $AC = 3$, and $\angle BAC = 90°$; then $AD^2 = AC^2 + CD^2$ (47. 1), $\therefore$ $AD = \sqrt{AC^2 + CD^2} = \sqrt{9 + 16} = \sqrt{25} = 5$; the body would therefore move over 5 yards by the joint action of the two forces.



Ex. 3.

Two equal forces, whose directions form an $\angle$ of 60°, act at the same time upon a body; What is the proportion of the compound force?

Let $AC = AB$, then $ABDC$ is a *Rhombus*; and as the $\angle BAC =$ 60°, the $\angle ACD$ must be equal to 120°; consequently each of the $\angle$s CAD, CDA, are 30°, ACD being an isosceles triangle. Hence $AD : AC$:: sin. 120° (or sin. 60°) : sin. 30° :: cos. 30° : sin. 30° :: (for sin. 30° $= \frac{1}{2}$ rad. $= \frac{1}{2}$, if radius $= 1$) $\sqrt{1 - \frac{1}{4}} : \frac{1}{2} :: \sqrt{\frac{3}{2}} : \frac{1}{2} ::$ $\sqrt{3} : 1$.

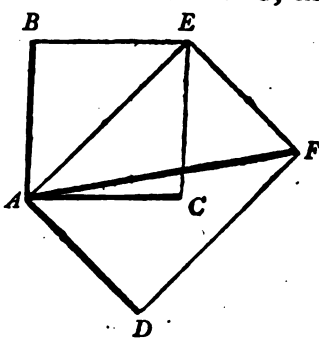


Ex. 4.

A body is acted upon at the same time by *three* equal forces; the second forms a right angle with the first, and the third forms an angle of 45° with the second; What is the relative magnitude of the compound force?

Let $AB = AC = AD$, $\angle BAC = 90°$, and $\angle CAD = 45°$; complete the parallelogram $ABEC$, and draw the diagonal AE; complete the parallelogram $AEFD$, and draw the diagonal AF; then AF will be the force compounded of

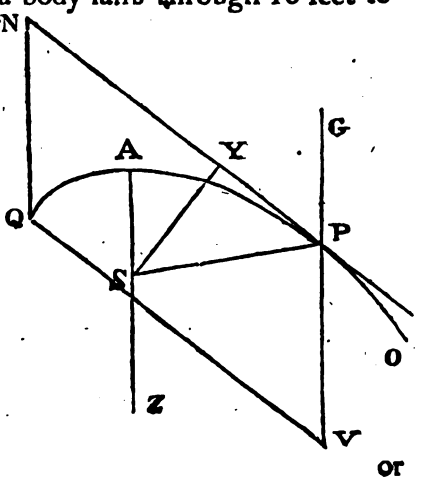
of the three. Since $AB = BE = EC = AC$, the figure $ABEC$ is a *square*, $\therefore \angle EAC = 45°$; but $\angle CAD = 45°$, $\therefore$ whole $\angle EAD = 90°$; hence $\angle AEF = 90°$. Now $AE^2 = AB^2 + BE^2 = 2AB^2$, and $AF^2 = AE^2 + EF^2 = AE^2 + AD^2 = 2AB^2 + AB^2$ (for $AE^2 = 2AB^2$ and $AD = AB$) $= 3AB^2$; hence $AF^2 : AB^2 :: 3 : 1$, or $AF : AB :: \sqrt{3} : 1$.



Ex. 5.

A body is projected with a velocity of 32 feet in a second, in a direction making an angle of 60° with a line drawn perpendicular to the horizon; It is required to find the *latus rectum* of the parabola which it describes, and its *velocity* at the vertex.

By Art. 9. Sect. V. a body falls through 16 feet to acquire a velocity of 32 feet per second, and by Art. 11. of the present section, the velocity in any point of the parabola is equal to that which it would acquire in falling through $\frac{1}{4}$th part of the parameter to that point; hence SP,



or

or $\frac{1}{4}$th of the parameter to the point P, is equal to 16 feet. Make therefore the angle $SPY = GPY = 60°$, and take $PS = 16$; then S will be the *focus* of the parabola. Now SP (16) : SY :: rad. : sin. 60° :: 1 : $\frac{\sqrt{3}}{2}$(a), $\therefore SY = 8\sqrt{3}$; but SP (16) : SY ($8\sqrt{3}$) :: SY ($8\sqrt{3}$) : $SA = \frac{192}{16} = 12$, or *latus-rectum* = 48; and the *velocity* at the vertex A = velocity acquired in falling through SA = velocity acquired in falling through 12 feet = velocity of 24 feet per second.

IX.

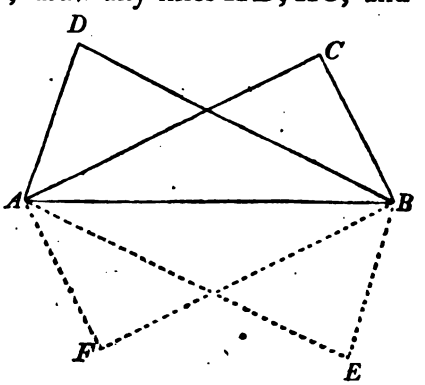
On the Resolution of Motion.

In the preceding section, we have shewn how a number of forces, whose quantities and directions are given, may be combined into a single one acting in some given direction; from which it evidently follows, that the motion of a body in any given direction may be considered as resulting from the united actions of several forces bearing certain given

(a) See Ex. 3. of this Section.

given relations to each other, as to quantity and direction, as will appear from the following investigation.

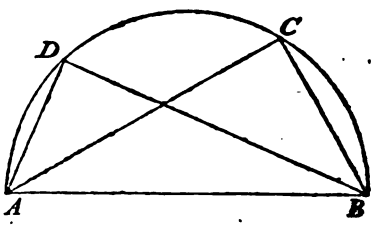
1. Let AB represent the quantity and direction of some given force; draw any lines AD, AC, and join DB, CB; complete also the parallelograms $ADBE$, $ACBF$. Since AB is the diagonal of two parallelograms whose adjacent sides are respectively AD, AE, and AC, AF,

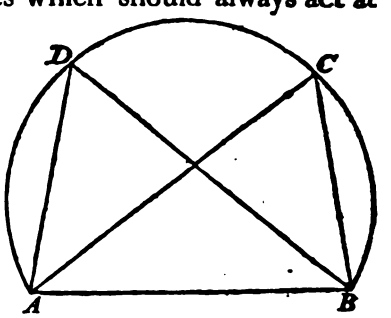


it may (by Art. 1. Sect. VIII.) be considered as resulting from the composition of two forces whose quantities and directions are represented either by AD, AE, or AC, AF, i. e. by AD, DB, or AC, CB; it is evident, therefore, that a given force AB may be resolved into as many *pairs of forces* as there can be triangles described *upon* a given right line AB, or parallelograms *about* it. And as the forces represented by AD, DB, or AC, CB, may also be resolved into other pairs of forces, it appears that, by proceeding in the same manner with the successive resulting forces, a given force may be resolved into an *unlimited* number of others acting in all possible directions.

2. But

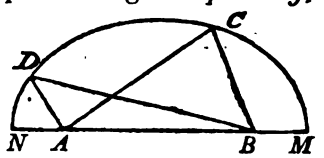
2. But the number of these forces will be *limited*, when the resolution of the given force takes place under certain restricting circumstances. For instance, if it were required to resolve a given force *AB* into different pairs of forces which should always act at *right-angles* to each other; then upon *AB* describe a *semicircle*, and then the number of pairs of forces into which the given force can be resolved will be limited to the number of triangles *ADB*, *ACB*, &c, which can be described upon the base *AB*, and have their vertices in the semicircumference *ADCB*; for besides these, no other *right-angled* triangle can be described upon the right line *AB*. Or, if it were required to resolve a given force *AB* into different pairs of forces which should always act at some *given angle* to each other, then upon *AB* (Euc. 33. 3.) describe a segment of a circle which shall contain an angle equal to the given angle, and the number of pairs of forces into which the given force can be resolved will be limited to the



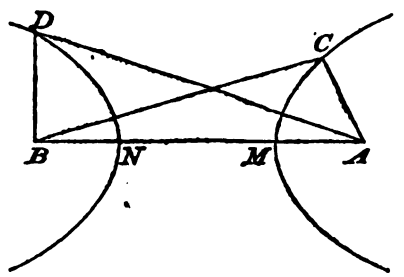


the number of triangles which can be described upon the base AB, and have their vertices in the circumference $ADCB$ of that segment.

3. The number of pairs of forces into which a given force may be resolved, so that the *sum* of the resolved forces should be equal to a *given quantity*, is determined by making AB the distance between the foci of an *ellipse* whose major axis MN is equal to the sum of the lines which represent the two forces into which the given force AB is to be resolved.



For in this ellipse take any points D, C, &c. and join BD, DA; BC, CA, &c.; then by the property of the ellipse $BD+DA$, $BC+CA$, and are all equal to the major axis MN; the number of pairs of forces, therefore, into which AB may be resolved, so that the sum of the resolved forces shall be a given quantity, is limited to the number of triangles which can be described upon AB, and have their vertices in the semi-elliptic arch $NDCM$. If it were required to resolve AB in such a manner that the *difference* of the resolved forces should be a given quantity, then make A, B, the foci of an *hyperbola* whose major axis MN is equal to that

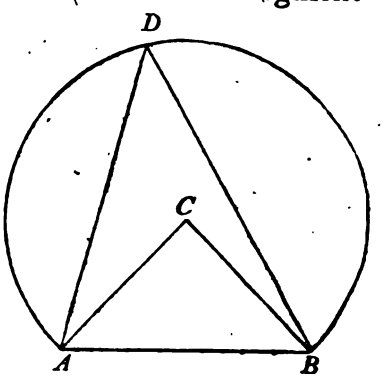


that given difference, and the vertices of the triangles whose sides represent the resolved forces would be found in the curve of that hyperbola.

Ex. 1.

A given force (a) is required to be resolved into different pairs of forces which shall act at an angle of 45° to each other; What is the radius of the circle, whose segment shall limit the number of the resolved forces according to Art. 2?

Let $AB=a$, and upon AB describe the segment of a circle which shall contain an angle ADB of 45°; then since $ADB=45°$, the angle ACB at the center = 90° (Euc. 20. 3.); hence (47. 1.) $AB^2 = AC^2 + CB^2 = 2AC^2$ (for $AC = CB$), $\therefore AC^2 = \frac{AB^2}{2} = \frac{a^2}{2}$, and the radius $AC = \frac{a}{\sqrt{2}}$.



Ex. 2.

To determine the radius of the circle when AB is required to be resolved into pairs of forces acting at any given angle A.

Let $AB=a$, $\angle ADB=A$; find center C, and join CA, CB; then $\angle ACB=2A$; $\angle CAB$ (CAB

(*CAB* being an *isosceles* triangle) $= \frac{1}{2}$ sum of $\angle$ s at base $= \frac{1}{2} \cdot 180 - 2A = 90^\circ - A =$ complement of A, $\therefore$ sin. $\angle CAB =$ cos. A; but AB (a) : BC :: sin. ACB ($2A$) : sin. CAB (cos. A), hence BC or

$$\text{radius} = \frac{a \times \text{cos. } A}{\text{sin. } 2A}.$$

Let $a = 10$, $A = 40^\circ$ } then radius $= \frac{a \times \text{cos. } A}{\text{sin. } 2A} = \frac{10 \times \text{cos. } 40^\circ}{\text{sin. } 80^\circ}$;

and log. $R =$ log. $10 +$ log. cos. $40^\circ -$ log. sin. $80 =$ $1 + 9.8842540 - 9.9933515 = 0.8909025 =$ log. 7.7786, $\therefore$ radius $= 7.7786$.

Ex. 3.

A given force (a) is resolved into different pairs of forces, the sum of which is equal to $2a$; What is the latus-rectum of the ellipse which limits the number of the resolved forces?

Since $MN = 2AB$ (Fig. in page 63), NA and BM are each equal to $\frac{1}{2}AB = \frac{1}{2}a$; $\therefore AM = \frac{3}{2}a$, and $NA \times AM = \frac{1}{2}a \times \frac{3}{2}a = \frac{3}{4}a^2$; but (by Prop. 2. Ellips.) semi-minor axis $= \sqrt{NA \times AM} = \sqrt{\frac{3}{4}a^2} = a \times \sqrt{\frac{3}{2}}$, $\therefore$ minor axis $= a\sqrt{3}$. Again, major axis ($2a$) : minor axis ($a\sqrt{3}$) :: minor axis ($a\sqrt{3}$) : latus-rectum $= \frac{3a^2}{2a} = \frac{3}{2}a$.

X.

On the relative Magnitude of Forces before and after their Composition or Resolution.

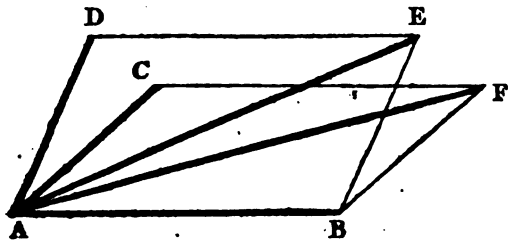
Having explained at some length the theory of the Composition and Resolution of Motion, we now proceed to shew the relation which the compound or resolved forces bear to those from which they are derived.

1. The most obvious consideration with respect to the *composition* of motion is, that, if two *equal* forces act upon a body in *contrary* directions, they will destroy each other's effects, and the body thus acted upon will remain *at rest*; or if *any two* forces act upon a body in the *same straight line*, then the effect (or, in other words, the *motion*) produced will be proportional to the *sum* or *difference* of those forces according as they act in the *same* or *opposite* directions. But if these forces act *obliquely* to each other, then the resulting force will be some intermediate quantity between that sum and difference, the magnitude of which will *increase* according as the angle of inclination between the directions of these forces is *diminished*[a].

2. This

(a) It is evident that the *smaller* the angle of inclination between two forces is, the more nearly will they *conspire together*, and consequently the *whole effect* produced will be *greater*; on the contrary, as the angle of inclination *increases*, the two forces will more strongly *oppose* each other, their *whole effect* therefore will keep *diminishing*.

2. This latter conclusion may also be drawn from the *geometrical* representation of the forces. Let two forces be represented by AB, AD, or by AB, AC, of which $AC = AD$; let the angle DAB be greater than the angle CAB, and complete the parallelograms $DABE$, $CABF$; then since DAB is *greater* than CAB, its supplement ABE must



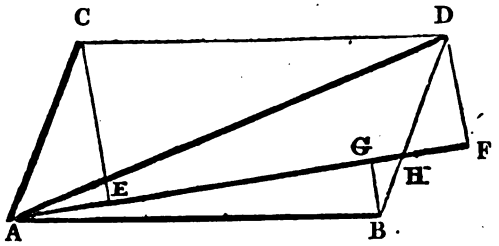
be *less* than ABF the supplement of CAB; hence in the triangles ABE, ABF, we have AB, BE equal to AB, BF, (for $BE = BF$ being opposite sides of a parallelogram) and the angle ABF greater than ABE, $\therefore$ (24. 1.) AF is greater than AE. Let $\angle CAB = 0$, then $AF = AC + CF = sum$ of forces; let $\angle CAB = 180°$, then $AF = CF - AC =$ *difference* of the forces; in all other cases AF is of some intermediate magnitude between $AC + CF$ and $CF - AC$, and keeps *increasing* as the angle CAB is *diminished*.

3. In the *composition* of forces which act obliquely to each other, some force is actually *lost*; for the sum of the forces *before* they are compounded together is represented by the two sides AD, DE of a triangle, and *after* composition by the third side AE.

The

The contrary happens with respect to the *resolution* of forces; for the two resolved forces being represented by the *two sides* of a triangle of which the given force is the *third*, the absolute quantity of the resolved forces must be *greater* than that of the given force.

4. But the effect of forces to produce motion in *given directions* is not changed by their composition



or resolution. For let the two forces AC, AB, and their equivalent AD, be resolved into the several forces AE, EC; AG, GB; and AF, FD; whereof AE, AG, AF are in the direction of some line AF given in position, and EC, GB, DF perpendicular to it; then by sim. $\triangle$ GHB, HDF, we have,

$GH : HF :: BH : HD$, & $GH+HF : HF :: BH+HD : HD$,
$\therefore GF : HF :: BD : HD$, or $GF : BD\ (=AC) :: HF : HD$.

But by sim. $\triangle^{s}$ AEC, HFD, $HF : HD :: AE : AC$;
Hence $GF : AC :: AE : AC$, $\therefore GF = AE$, and $AF = AG + AE$; i.e. that part of the *compound* force AD which acts in the direction AF is equal to such parts of the *resolved* forces as act in the same direction.

Again,

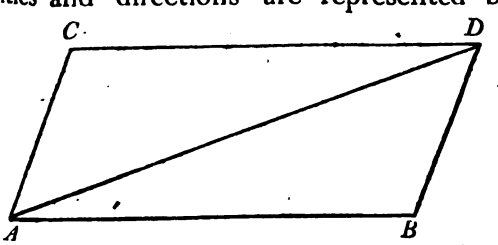
Again, by sim. Δ^s, $BG : DF :: BH : HD$;
$\therefore BG + DF : DF :: BD$ or $AC : HD$,
or $BG + DF : AC :: DF : HD :: CE : AC$.

Hence $BG + DF = CE$, $\therefore DF = CE - BG$; but since CE, BG represent forces acting in *opposite* directions to each other, their effect in the direction CE perpendicular to AF will be represented by their difference $CE - BG$; $\therefore$ since $DF = CE - BG$, the part of the *compound* force which is perpendicular to AF is equivalent to such parts of the *resolved* forces AC, AB as are perpendicular to AF.

XI.

On the Nature of the Forces, which, by their joint Action, would keep a Body at rest.

1. It has already been observed, that if a body be acted upon by two equal and contrary forces, it will remain at rest; it has also been proved, that if a body be acted upon by two forces whose quantities and directions are represented by the



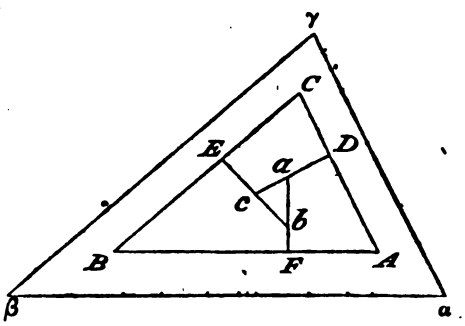
two sides AC, AB of the parallelogram $ABDC$, or by the two sides AC, CD, of the triangle ACD, the diagonal or third side AD will represent a force equivalent

equivalent to them both. Suppose now, that at the same time that the two forces represented by *AC, AB,* or *AC, CD,* are applied to a body placed at *A,* a third force represented in quantity and direction by the line *DA* (equal and opposite to *AD*) should also be applied to the body, as it would in this case be acted upon by two *equal and contrary* forces, it would remain at *rest.* A body therefore acted upon at the same time by three forces represented in quantity and direction by *AC, CD, DA,* i. e. *by the three sides of a triangle taken in order,* will remain at rest.

2. The *converse* of this proposition must also be true, viz. "that if a body be kept at rest by three forces, those three forces will be represented by the three sides of a triangle formed by lines drawn in their respective directions." For suppose a body to be kept at rest by three forces, and that *AC, CD* represent the quantities and directions of *two* of those forces, then the *compound* force arising from those two forces will be represented by the line *AD*; if, therefore, the *third* force be not represented in quantity and direction by the line *DA,* equal and opposite to *AD,* some motion must ensue, which is contrary to the supposition. Whenever, therefore, a body is kept at rest by three forces, if a triangle be drawn, whose sides are respectively in the direction of those forces, those sides will represent the quantity and direction of the several forces thus acting upon the body.

3. But

3. But the proportion of the three forces which thus keep a body at rest will be represented by the three sides of any triangle drawn *parallel* or *perpendicular* to the sides of the triangle which are in the directions of the forces. For let the triangle ABC be that whose sides are drawn in the direction of the three forces, then the triangle $\alpha\beta\gamma$ (whose sides are *parallel* to AB, BC, CA) and the triangle abc (whose sides are *perpendicular* to AB, BC, CA) being each of them similar[a] to the triangle ABC, must have their sides $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$, or ab, bc, ca, respectively proportional to the three sides AB, BC, CA, which represent the quantity and direction of the forces acting upon a body.

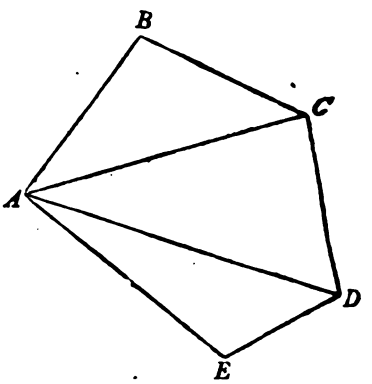


4. Since $AB : BC :: \sin. \angle BCA : \sin. \angle BAC$; $BC : CA :: \sin. \angle BAC : \sin. ABC$; and $CA :$ AB

(a) Since the sides of the triangle $\alpha\beta\gamma$ are respectively parallel to the sides of the triangle ABC, it is evident that the angles $\alpha\beta\gamma$, $\beta\gamma\alpha$, $\gamma\alpha\beta$ are respectively equal to ABC, BCA, CAB; with respect to the triangle abc, since the $\angle_s$ at D, E, F are *right*-angles, we have $DaF + DAF = 180_0$, also $DaF + bac = 180^0$, $\therefore DAF = bac$; and in the same manner it appears that $EBF = abc$, and $ECD = acb$.

AB :: sin. $\angle ABC$: sin. $\angle BCA$, any *two* of these forces which keep a body at rest are to each other *inversely* as the sines of the angles which their respective directions make with the *third* force; and *conversely*, "if a body be acted upon by three forces, any two of which are to each other inversely as the sines of the angles which their directions make with the third force, it will in that case remain at rest."

5. A body will also be kept at rest if it be acted upon by any number of forces which are represented in quantity and direction by the sides of a *polygon* taken in order. For let a body be acted upon by any number of forces represented by the sides AB, BC, CD, DE of the polygon $ABCDE$; then (by Art. 5. Sect. VIII.)

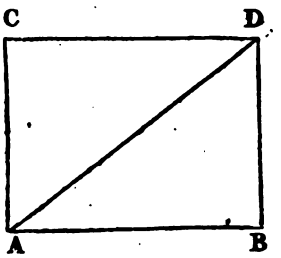


these forces compounded together will be represented in quantity and direction by the remaining side AE; if, therefore, at the same time that the body is acted upon by the forces AB, BC, CD, DE it is also acted upon by another force represented in quantity and direction by EA (equal and opposite to AE), it will remain at rest; i. e. "a body acted upon by forces represented in quantity and direction by

by the sides AB, BC, CD, DE, EA of the polygon $ABCDE$, taken in the order, will remain at rest." The *converse* of this proposition may also be established by the same mode of reasoning as that made use of in Art. 2.; viz. "If a body be kept at rest by any number of forces, those forces will be represented, in quantity and direction, by the sides of a polygon formed by the intersection of lines drawn in the directions in which they respectively act."

Ex. 1.

A body is acted upon by two forces a and b, which are at right angles to each other; it is required to find the magnitude and direction of a *third* force, which shall keep the body *at rest*.



Let $AC=a$, $AB=b$, $\angle CAB=90^\circ$ } Complete the parallelogram $ABDC$, and join AD; then the two forces acting upon the body may be represented by AC, CD; consequently DA (the third side of the triangle ACD) will represent the force which shall keep the body *at rest*. Now $DA=\sqrt{AC^2+CD^2}=\sqrt{a^2+b^2}$ for the *magnitude* of the force; and sine of $\angle CAD$: sine of $\angle CDA$ $(=DAB)$:: CD : CA :: b : a; $\therefore$ the *direction* DA of the *third* force divides the $\angle CAB$ into two angles, whose *sines* are to each other as a : b.

Ex. 2.

A body, acted upon by two forces, is kept at rest by a third force (a), whose direction divides the angle contained between the directions of the two former into the given angles A and B; What is the *magnitude* of those two forces?

Let AC, AB be the two forces (Fig. in page 69); complete the parallelogram $ABDC$, then DA ($=a$) is the *third* force; let $\angle CAD = A$, $\angle DAB = B$, then $\angle ACD = 180^\circ - \overline{A+B}$. Now

$AC : AD\ (a) :: \sin. CDA \text{ or } DAB\ (B) : \sin. ACD\ (180^\circ - \overline{A+B},)$

& $AB : AD\ (a) :: \sin. ADB \text{ or } CAD\ (A) : \sin. ACD\ (180^\circ - \overline{A+B},)$

Hence $AC = \dfrac{\sin. B \times a}{\sin. 180^\circ - \overline{A+B}}$, and $AB = \dfrac{\sin. A \times a}{\sin. 180^\circ - \overline{A+B}}$.

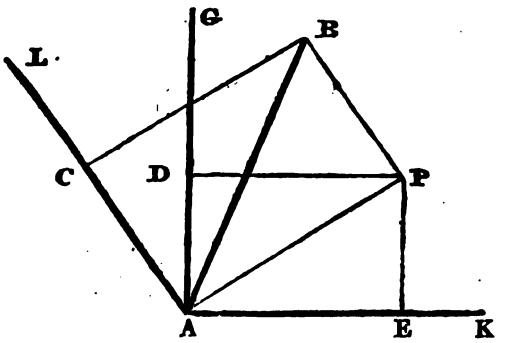
XII.

On the Composition and Resolution of Motion arising from the Operation of Forces acting in different Planes.

In estimating the effects produced by the composition and resolution of forces, we have hitherto considered them as acting in the *same plane*; we proceed now to the solution of the problem, by means of which we are enabled to determine the motion of a body resulting from the operation of any number of forces acting in different planes.

1. Let AK, AG be two straight lines drawn at right

right angles to each other in the same plane, and let AL be drawn at right angles to that plane, and, consequently, at right angles to each of the lines AK, AG (Euc. B. 11. Def. 3.) Suppose AB to represent the quantity and direction of a force acting upon a body at A; let fall the perpendicular

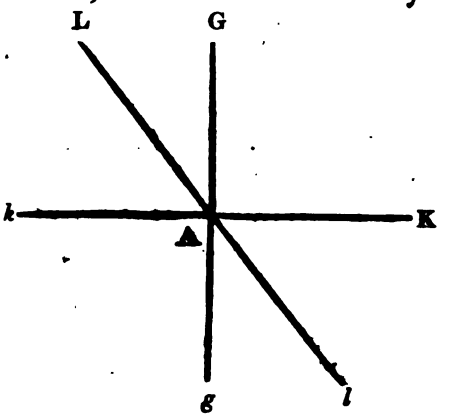


BP upon the plane passing through AK, AG; join AP, and complete the parallelogram $APBC$. From P draw PD parallel to AK, and PE parallel to AG. Since AB is the diagonal of the parallelogram $APBC$, the force represented by AB is resolved into two others AC, AP equivalent to it; and since AP is the diagonal of the parallelogram $ADPE$, the two AD, AE are equivalent to AP. Hence the given force AB is resolved into *three* others AC, AD, AE, in the direction of the three straight lines AL, AG, AK, which are *at right angles to each other, and issue from the point A.*

2. Produce LA to e, GA to g, and KA to k; so that the three lines (or axes) Ll, Gg, Kk, shall cut each other at right angles in the point A; then it

it is evident that the directions of all the forces, which can possibly act upon a body at the point *A*, will fall within one or other of the eight solid angles formed by the intersection of three planes cutting each other at right angles, and passing through the axes *Ll*, *Gg*; *Ll*, *Kk*; *Gg*, *Kk*, respectively; and from what has just been shewn, each of those forces may be resolved into *three* others, in the directions,

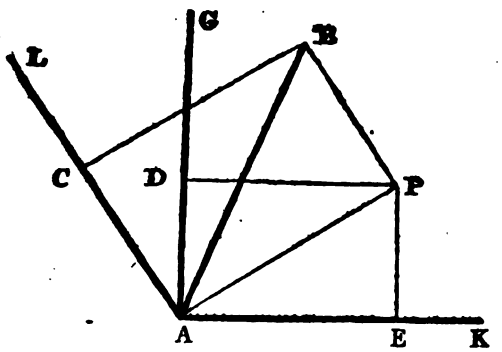
AL, *AG*, *AK*;
AL, *AG*, *Ak*;
AL, *Ag*, *Ak*;
AL, *Ag*, *AK*;
or *Al*, *AG*, *AK*;
Al, *AG*, *Ak*;
Al, *Ag*, *Ak*;
Al, *Ag*, *AK*;



according to the solid angle in which it is included. Thus, then, all the forces which can possibly act upon a body at the point *A*, may be resolved into others acting along the three axes *Ll*, *Gg*, *Kk*; for the forces acting in the directions *AL*, *Al*; *AG*, *Ag*; *AK*, *Ak*, respectively, are merely forces acting in *opposite* directions in the same straight line. Our next business, therefore, is to estimate the *magnitude* of these resolved forces.

3. We shall begin with calculating the relative value of the three forces *AC*, *AD*, *AE*, into which a *single* force *AB* may be resolved. For which purpose

purpose let $AB=d$; radius $=1$; $\angle BAP$ (which measures the inclination of AB to the plane passing



through AK, $AG)=a$; $\angle PAE$ (which measures the angular position of AB with respect to the fixed line $AK)=\alpha$.

Now,

$AB\ (d) : AP$:: radius (1) : cos. BAP (cos. a), $\therefore AP=d\times$ cos. a.

To find the magnitude of AC, AD, AE, we have,

$AB\ (d)$: BP or AC :: rad. (1) : sin. BAP (sin. a),
$\therefore AC=d\times$ sin. a.

$AP\ (d\times$ cos. $a)$: PE or AD :: rad. (1) : sin. PAE (sin. α),
$\therefore AD=d\times$ cos. $a\times$ sin. α.

$AP\ (d\times$ cos. $a)$: AE :: rad. (1) : cos. PAE (cos. α),
$\therefore AE=d\times$ cos. $a\times$ cos. α.

4. In order to facilitate the computation of the value of these quantities as applied to *any number* of forces, let d, d', d'', &c. represent the *magnitudes*; a, a', a'', &c. the *angles of inclination*; α, α', α'', &c. the *angular distance* from the fixed line AK; of the forces acting from the point A in directions *above* the plane passing through the axes Gg, Kh

(Fig.

(Fig. in p. 76), and δ, δ', δ'', &c. the *magnitudes*; b, b', b'', &c. the *angles of inclination*; β, β', β'', &c. the *angular distance* from AK; of the forces acting *below* that plane. By resolving these several forces in the same manner as the force AB has just been resolved, the effective parts of them acting along the *three axes* Ll, Gg, Kk, may be calculated according to the following method; viz.

$d \times \sin. a + d' \times \sin. a' + d'' \times \sin. a'' + \&c. - \delta \times \sin. b - \delta' \times \sin. b' - \delta'' \times \sin. b'' - \&c.$

will represent the whole quantity of force acting along the axis Ll, reckoned in the direction AL;[a]

$d \times \cos. a \times \sin. \alpha + d' \times \cos. a' \times \sin. \alpha' + \&c. + \delta \times \cos. b \times \sin. \beta + \delta' \times \cos. b' \times \sin. \beta' + \&c.$

(making the sines of $\alpha, \beta, \alpha', \beta'$, &c. *positive* or *negative*, according as the angles themselves are *less* or *greater* than 180°) will express the whole quantity of force acting along the axis Gg, reckoned in the direction AG; and,

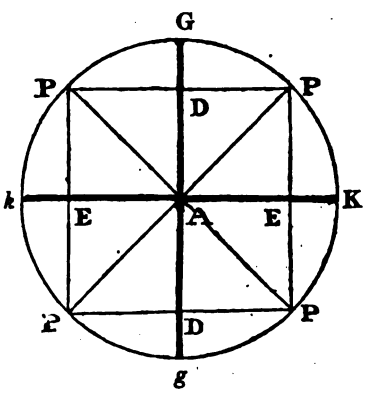
$d \times \cos. a \times \cos. \alpha + d' \times \cos. a' \times \cos. \alpha' + \&c. + \delta \times \cos. b \times \cos. \beta + \delta' \times \cos. b' \times \cos. \beta' + \&c.$

(making the cosines of $\alpha, \beta, \alpha', \beta'$, &c. *positive* for all angles between 0 and 90°; or between 270° and 360°; and *negative* for all angles between 90° and 270°) will

(a) It is evident, that to find the whole quantity of force acting in direction AL, to carry the body out of the plane which passes through Gg, Kk, we must *subtract* the sum of the forces acting in direction Al from the sum of those which act in direction AL.

will express the whole quantity of force acting along the axis Kk, and reckoned in the direction AK.[b]

By way of explanation of these trigonometrical expressions, it may be observed, that, as *the angles of inclination of the forces to the plane passing through the axes* Gg, Kk, can never exceed 90°, the sines and cosines of those angles will all be *positive*; but as *the angles of position of the forces with respect to the line* AK may vary through all degrees of magnitude between 0 and 360°, *their* sines and cosines will be either *positive* or *negative*, according to the *different quadrants* to which they belong. To illustrate this; with center A describe a circle round the axes Gg, Kk, and let AP represent the direction of any one of the forces d, d', d'', &c. or δ, δ', δ'', &c. when reduced to the plane of this circle. Suppose now the angular position of AP with respect to AK to be measured on the



(b) In computing the magnitude of the forces acting along the axes Gg, Kk, it should be recollected, that although the signs of all the terms as they stand here are *positive*, yet, when the proper signs

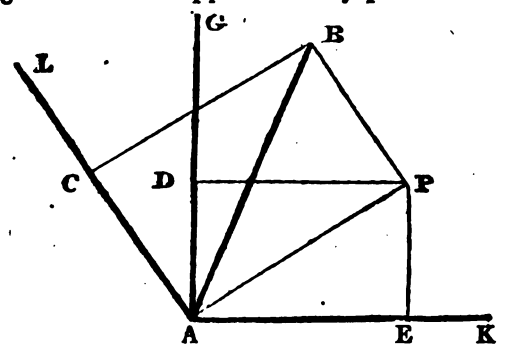
the circumference of this circle, beginning from the point K; then through the two quadrants KG, Gk the *sine* PE of the angle PAK is *positive*, and through the two quadrants kg, gK it is *negative*; with respect to the *cosine* AE of the $\angle PAK$, it is *positive* for the quadrant KG, *negative* for the two quadrants Gk, kg, and *positive* again for the quadrant gK.[a] Having thus ascertained the signs of these trigonometrical quantities, it only remains to collect them into three several sums corresponding to the three axes Ll, Gg, Kh; the resulting quantities will represent the magnitude of *three forces* (issuing from A in the directions of those axes) by the *joint action* of which the motion of the body placed at A will be ultimately determined.

6. We have now so far advanced in the solution of this somewhat intricate problem, as to have reduced all the forces which can possibly act upon a body at A to *three others* acting at right angles to each other. To shew how these three forces may be

signs are annexed to the sines and cosines of the angles of position, α, β, α', β', &c. these expressions will represent the *difference* which arises from subtracting the forces in direction Ag from those in direction AG, and the forces in direction Ak from those in direction AK.

(a) For this + and − variation of the sine and cosine of an angle through the different quadrants of the circle, see the Table in the Second Lecture of my treatise on Plane Trigonometry.

be *compounded together*, we need only attend to the manner in which the given force AB was resolved into three others, AC, AD, AG, at our first setting out. For suppose a body placed at A, and



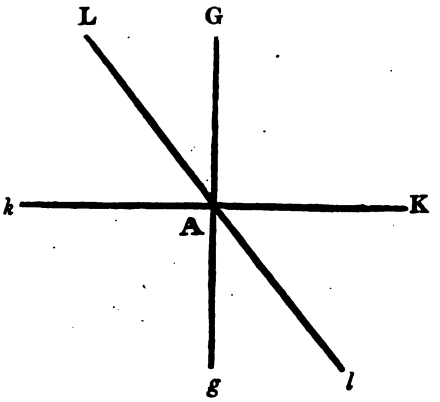
acted upon by three forces whose quantities and directions are represented by the three lines AC, AD, AE, which meet each other at right angles in the point A; then *reversing* the operation in Art. 1. the two forces AD, AE, compounded together, will be represented by the diagonal AP of the parallelogram $ADPE$; and the two forces AE, AP, compounded together, will be represented by AB, the diagonal of the parallelogram $APBC$; the *three* forces, thus acting upon the body, are consequently reduced to *one* (AB) equivalent to them all[b]. For the

(b) From the construction of the figure, it appears that AB is the diagonal of a right-angled *parallelopipedon*, whose base is the parallelogram $ADPE$ formed by the two lines AD, AE, and whose height is AC, the line representing the third force.

the purpose of finding the actual quantity and direction of the force represented by AB, let AC or $BP=x$, AD or $PE=y$: AE or $DP=z$; then (47. 1.) $AB^2=BP^2+AP^2=$ (for $AP^2=AD^2+DP^2$) $BP^2+AD^2+DP^2=x^2+y^2+z^2$, $\therefore AB=\sqrt{x^2+y^2+z^2}$ for the *quantity* of the compound force. With respect to its *direction*, we have (supposing rad. = 1) $AP\ (\sqrt{y^2+z^2}) : PE\ (y)$:: rad. (1) : sin. $PAE=\frac{y}{\sqrt{y^2+z^2}}$; this gives the position of AP with respect to the line AK. Again, $AB\ (\sqrt{x^2+y^2+z^2}) : BP\ (x)$:: rad. (1) : sin. $BAP=\frac{x}{\sqrt{x^2+y^2+z^2}}$, which gives the inclination of AB to the plane passing through AG, AK; on supposition therefore that x, y, z are determined quantities the *magnitude* of the force AB, its *position with respect to the fixed line* AK, and its *inclination to the plane passing through* AG, AK, are actually known.

7. For the purpose of applying these expressions to ascertain the quantity and direction of a single force *equivalent to the forces* d, d', d'', &c. δ, δ', δ'', &c. acting upon the body at A; let the sum of all the forces acting *along the axis* Ll (as exhibited in Art. 4.) $=\pm x$; the sum of those acting *along the* axis $Gg=\pm y$; and the sum of those acting *along the axis* $Kk=\pm z$; then the *quantity* of this single force will be represented by

by $\sqrt{x^2+y^2+z^2}$; its *position with respect to the fixed line AK* will be measured by an angle whose sine is $\frac{\pm y}{\sqrt{y^2+z^2}}$; the particular quadrant to which it belongs being determined by the signs of y and z; its *inclination to the plane, passing through Gg, Kk,* will be in an angle whose sine is $\frac{\pm x}{\sqrt{x^2+y^2+z^2}}$, and it will fall *above* or *below* that plane, according as x is *positive* or *negative*.



8. By way of conclusion, it may be observed, that if $x^2=0$, then the motion of the body will be in the plane passing through Gg, Kk; its *quantity* will be represented by $\sqrt{y^2+z^2}$, and its angular position with respect to AK by an angle whose sine is $\frac{\pm y}{\sqrt{y^2+z^2}}$. If $y^2=0$, then the motion of the body will be in the plane passing through Ll, Kk; its *quantity* will be $\sqrt{x^2+z^2}$; and its *inclination* to the plane passing through Gg, Kk, in an angle whose sine is $\frac{\pm x}{\sqrt{x^2+z^2}}$. If $z^2=0$, then its motion will be in the

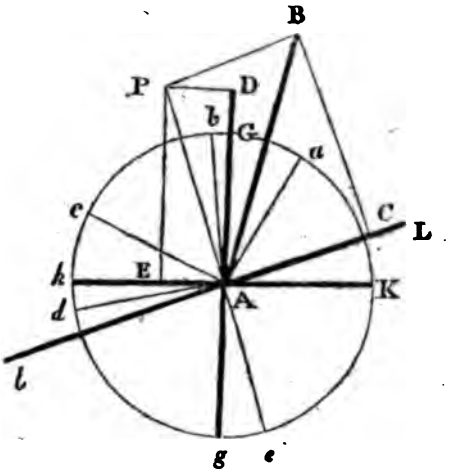
the plane passing through Ll, Gg; its *quantity* will be $\sqrt{x^2+y^2}$, and its *inclination* to the plane passing through Gg, Kk, will be in an angle whose sine is $\frac{\pm x}{\sqrt{x^2+y^2}}$. If *two* of these quantities (x^2, y^2, or z^2) should become equal to 0, then the motion of the body will be along the axis belonging to the *third*; and if *all three* become equal to 0 (or $\sqrt{x^2+y^2+z^2}=0$) then all the forces acting upon the body at A are in a state of *equilibrium*, and the body will remain *at rest*.

EXAMPLE.

A body is acted upon by five forces, which are to each other as the numbers 7, 6, 5, 3, 2; *the directions of the three former are* above, *and of the two latter* below, *a plane given in position, and passing through the body. The forces acting* above *the plane are inclined to it in angles of* 25°, 45°, *and* 60°; *those acting* below, *in angles of* 50° *and* 85°, *respectively. The angles of position of these forces, with respect to a certain fixed line in that plane, taken in the order of the numbers* 7, 6, 5, 3, 2, *are* 60°, 95°, 150°, 200°, *and* 280°. *It is required to find the* quantity *of a force equivalent to them all, together with its* direction, *both as to the plane, and to the fixed line drawn in that plane.*

Let A be the place of the body, $abcde$ a circle drawn

drawn in the plane passing through the body, *Gg*, *Ll* two axes drawn at right angles to each other in that plane, and *Ll* the axis at right angles to *Gg*, *Kk*, and consequently at right angles to the plane. Let the angular position of the forces be reckoned from the fixed line *Kk*; then if *Ka*, *Kb*, *Kc*, *Kd*, *Ke* are taken arcs of 60°, 95°, 150°, 200°, and 280°, respectively, *Aa*, *Ab*, *Ac*, *Ad*, *Ae* will be the angular position of the forces, when reduced to the plane *abcde*.



Calculating the quantity of force acting along the axes *Ll*, *Gg*, *Kk*, according to the rule laid down in Art. 4, we shall have $7 \times \sin. 25^\circ + 6 \times \sin. 45^\circ + 5 \times \sin. 60^\circ - 3 \times \sin. 50^\circ - 2 \times \sin. 85^\circ = 7 \times .4226 + 6 \times .7071 + 5 \times .8660 - 3 \times .7660 - 2 \times .9922 = 7.2406$ for the force acting *along the axis Ll*, which being *positive*, take *AC* (in the direction *AL*) = 7.2046, and it will represent the quantity of force impelling the body *upwards* from the plane *abcde*.

For

For the sum of the forces acting *along the axis Gg*, we have $7 \times \cos. 25^\circ \times \sin. 60^\circ + 6 \times \cos. 45^\circ \times \sin. 95^\circ + 5 \times \cos. 60^\circ \times \sin. 150^\circ - 3 \times \cos. 50^\circ \times \sin. 200^\circ - 2 \times \cos. 85^\circ \times \sin. 280^\circ = 7 \times .9063 \times .8660 + 6 \times .7071 \times .9961 + 5 \times .5000 \times .5000 - 3 \times .6427 \times .3420 - 2 \times .0871 \times .9848 = 9.7388$; which being also *positive*, take AD (in the direction AG) $= 9.7388$, and it will represent the quantity of force acting in that direction.

Lastly, $7 \times \cos. 25^\circ \times \cos. 60^\circ - 6 \times \cos. 45^\circ \times \cos. 95^\circ - 5 \times \cos. 60^\circ \times \cos. 150^\circ - 3 \times \cos. 50^\circ \times \cos. 200^\circ + 2 \times \cos. 85^\circ \times \cos. 280^\circ = 7 \times .9063 \times .5000 - 6 \times .7071 \times .0871 - 5 \times .5000 \times .8660 - 3 \times .6427 \times .9396 + 2 \times .0871 \times .1736 = -1.1423$; which being *negative*, take AE (in direction Ak) $= 1.1423$, and it will represent the quantity of force acting *along the axis Kk*.

Complete now the parallelogram $ADPE$, and join AP; then AP will be the direction of the compound force in the plane $abcde$; complete also the parallelogram $APBC$, and join AB; then AB will represent the *quantity* and *direction* of the force resulting from *the composition of all the forces acting upon the body at A*.

To find the actual magnitude and direction of AB, we must have recourse to the process adopted in Art. 7. for in this case we have $x = 7.2406$, $y = 9.7388$, $z = -1.1423$; $\therefore \sqrt{x^2 + y^2 + z^2} = \sqrt{148.574}$

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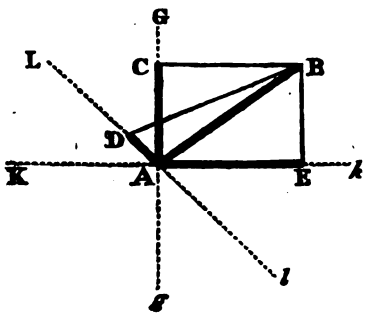
$= 12.18 = AB$. Also, sin. $PAK = \frac{y}{\sqrt{y^2+z^2}} = \frac{9.7388}{\sqrt{96.148}} = .99324$; $\therefore$ PAK is an angle whose *sine* $= .99324$, but whose *cosine* (z) is *negative*, i.e. an angle of 96°. 40′. Finally, sin. $BAP = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{7.2406}{12.18} = .5944 =$ sin. 36°. 28′. Hence AB is inclined to the fixed line Kk in an angle of 96°. 40′, and its inclination *above* the plane *abcde* is in an angle of 36. 28′.

XIII.

We have thus shewn the method of resolving all the forces which can possibly act upon a body into the direction of "*three axes*" at right angles to each other. If this resolution takes place in the direction of each axis separately, without reference to *the plane passing through the other two*; and if the forces at the same time be in a state of *equilibrium*; then the solution of the problem becomes somewhat more simple, as will appear, from the following investigation.

1. Let AB represent the quantity and direction of any given force, and let Gg, Ll, Kk be three axes cutting each other at right angles at the point A; from B let fall the perpendiculars BC, BD,

BD, BE upon *AG, AL, AK* respectively; then *AC, AD, AE* (which are the *cosines* of the angles



BAC, BAD, BAE) represent the parts of the given force which are in the direction of the axes *Gg, Ll, Kk.*

2. Hence if P, P', P'', &c. be *any number* of forces acting in the point *A*; and if $\alpha, \alpha', \alpha''$, &c. be the angles which they make with the *axis* Gg; β, β', β'', &c. the angles which they make with the *axis* Ll; $\gamma, \gamma', \gamma''$, &c. the angles which they make with the axis Kk, respectively, and *radius* $= 1$; then, when these forces are in *equilibrio* with each other at the point *A*,

$$P \times \cos.\alpha + P' \times \cos.\alpha' + P'' \times \cos.\alpha'' + \&c. = 0$$
$$P \times \cos.\beta + P' \times \cos.\beta' + P'' \times \cos.\beta'' + \&c. = 0$$
$$P \times \cos.\gamma + P' \times \cos.\gamma' + P'' \times \cos.\gamma'' + \&c. = 0$$

will be three independent equations which determine the relation of these several quantities to each

each other at the time the equilibrium takes place[a].

3. If the forces act in the *same plane* (viz. the plane passing through the axes Gg, Kk), then, cos. β, cos. β', cos. β'', &c. $= 0$, and cos. γ, cos. γ', cos. γ'', &c.[b] will be equal to sin. α, sin. α', sin. α'', &c. $\therefore$ these three equations are reduced to two, viz.

$$P \times \text{cos. } \alpha + P' \times \text{cos. } \alpha' + P'' \times \text{cos. } \alpha'' + \&c. = 0$$
$$P \times \text{sin. } \alpha + P' \times \text{sin. } \alpha' + P'' \times \text{sin. } \alpha'' + \&c. = 0$$

which will determine the relation of the quantities to each other when P, P', P'', &c. are in a state of equilibrium.

4. As there are *three* independent equations in Art. 2, and *two* in Art. 3; the magnitude or position

(a) For each force may be resolved into three others in the direction of the *three axes* Gg, Ll, Kk, of which the parts *effective* in the direction of *each* axis will be to the *force itself* :: cosine of the angle which it makes with the axis : *radius* (1); thus, the part effective along the axis Gg : P :: cos. α : 1, or *part effective* $= P \times$ cos. α; and so of the rest. And since the forces are in *equilibrio*, the *sum of the resolved forces* in the direction of *each* axis must be equal to 0; from which the nature of these equations is manifest.

(b) For since AB is the plane passing through the axis Gg, Kk, and $\angle BAC = \alpha$, $\therefore \angle BAE = 90^0 - \alpha$; hence AC = cos. BAC = cos. α, and AE = cos. BAE = cos. $\overline{90^0 - \alpha}$ = sin. α; and so of the rest.

position (with respect to *one* of the three axes) of any *three* of the forces P, P', P'', &c. in the *former* case, and of any *two* of the *latter*, may be considered as undetermined quantities, whose values are to be ascertained from the relation which subsists between the remaining *given* quantities at the time the equilibrium takes place. For instance, let there be three forces P, P', P'', acting in *different planes*, and let every thing be given except *the magnitude of the two last*, and *the position of the first* with respect to the axis Kk; then, if $P' = x$, $P'' = y$, cos. $\gamma = z$,

$$P \times \cos.\, \alpha + x \times \cos.\, \alpha' + y \times \cos.\, \alpha'' = 0$$
$$P \times \cos.\, \beta + x \times \cos.\, \beta' + y \times \cos.\, \beta'' = 0$$
$$Pz \ldots + x \times \cos.\, \gamma' + y \times \cos.\, \gamma'' = 0$$

are three equations, from which such values of x, y, z (or P', P'' and cos. γ) may be deduced, as shall produce an equilibrium of the three forces.

END OF THE LECTURE.

Lecture IV.

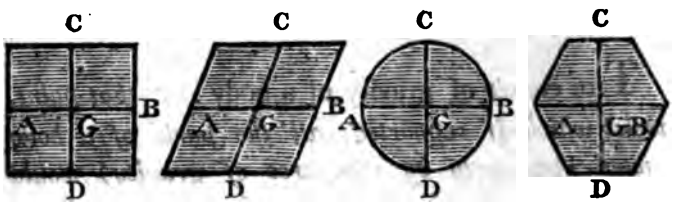
ON THE CENTER OF GRAVITY.

THE center of gravity of a body is "*that point, about which, if supported, all the parts of a body (acted upon only by the force of gravity) would balance each other in any position.*" In order therefore to ascertain this point, it will be necessary to resolve a body into its *constituent parts;* and then find two lines, about *each* of which (if supported) these parts will balance each other in all positions: the *common intersection* of those two lines will be the *center of gravity* required. In bodies of a *regular form and uniform texture* this is very easily effected, but the difficulty increases as the nature or shape of the body becomes more complex. We shall begin with shewing the method of finding the center of gravity of a body, or of a *system* of bodies, in a few familiar instances.

XIV.

On the Method of finding the Center of Gravity of a Body or System of Bodies.

1. In regular plane figures, such as *squares, parallelograms, circles, polygons inscribed in circles*, &c. &c. the center of *gravity* is the same with the center of *magnitude*. For instance, let the annexed figures represent thin laminæ of matter of an uniform density, and let them be



divided respectively into two equal parts by the straight lines AB, CD.(a) Conceive now each of these laminæ to be resolved into *lines of particles* equal and parallel to AB, there will then be the same quantity of matter similarly disposed on each side

(a) In the *square* and *parallelogram* this will be done by bisecting any two opposite sides, and joining the points of bisection; in the *circle*, by drawing two diameters at right angles to each other; in the *hexagon*, by drawing AB to two opposite angles, and bisecting two opposite sides by CD; &c. &c.

side of AB; if, therefore, AB be supported, the parts ACB, ADB will balance themselves about it; the center of gravity will consequently be in the line AB. For the same reason, because all lines drawn parallel to AB are bisected by CD, the center of gravity will also be in the line CD; it must therefore be in their common intersection G.

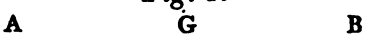
2. In the same manner it might be shewn, that the center of gravity of the regular solids, such as the *cube*, *parallelopipedon*, *cylinder*, *sphere*, &c. is the same with the center of magnitude. For each of these solids might be divided into two equal and similar parts by planes passing through it in three different directions; the intersection of *two* of these planes would be a *right line*, and the intersection of that line with the *third* plane would be the center of gravity of the solid.[b]

3. Suppose now A and B (Fig. 1. page 94) to be two

[b] Let the *three first* figures in the opposite page represent respectively the section of a *cube*, *parallelopiped*, and *sphere*, cut through their middle; then may the line CD represent the intersection of a plane at right angles to $ACBD$ (and AB the intersection of a *third* plane) cutting these solids in a similar manner; the point G will therefore be the center of gravity. In the *cylinder*, it is evident that the center of gravity will be in the point which *bisects its axis*.

two *equal* particles of matter connected together by the inflexible rod AB void of gravity; bisect AB in G, then G will be the *common center of gravity* of A and B; for it is evident, that if G be supported, the two particles will balance themselves about it. The *pressure* upon G will be equal to the *weight* of the particles A and B, and this pressure does not at all depend upon the *length* of the line AB; it will therefore be the same whether the particles be placed at A and B, or a particle equal to $A+B$ be placed at G. The same may be said with respect to the particles A, B, C, D, &c. (in Fig. 2.) which are disposed *uniformly* along the inflexible rod AN void of gravity; viz. that the pressure of A *and* N is the same as if $A+N$ was placed at G; of B *and* M the same as if $B+M$ was placed at G; &c. &c. and that the *whole pressure* of the particles A, B, C, D, &c. is the same as if $A+B+C+D+E+F$, &c. was placed at G.

Fig. 1.



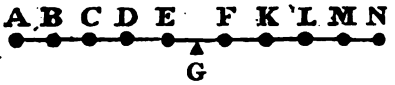


Fig. 2.

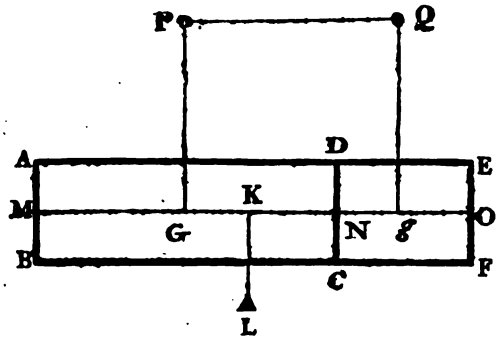
4. This reasoning might be extended to the *lines of particles* composing the *laminæ* in Art. 1; for the particles A, B, C, D, &c. (Fig. 2.) may be increased in number till they become *contiguous* to each other, and the effect is the same whether we consider them as connected together by an inflexible rod

rod void of gravity, or actually united together by the *power of cohesion.* Supposing CD therefore to be supported (see Figures in Art. 1.) the pressure upon it will be the same as if all the particles contained in the lines parallel to AB were incumbent upon it; and supposing the point G only to be supported, the pressure will be the same as if the particles thus collected in CD were incumbent upon it; the pressure of the lamina $ACBD$ upon the center of gravity is therefore the same as if all the matter contained in it were incumbent upon G. The same mode of demonstration might be applied to the laminæ composing the regular solid bodies in Art. 2; so that it may be asserted as a *general truth,* that when a body of this kind is supported by a prop placed under its center of gravity, the pressure will be the same whether the whole quantity of matter be uniformly diffused through the space occupied by the body, or whether it be all as it were *concentrated* in that center of gravity.

5. For the purpose of investigating the theorem for finding the common center of gravity of two *unequal* bodies, let $ABCD$, $CDEF$ (Fig. in page 96) represent the sections of two cylinders of *uniform density* and of the *same diameter,* whose *axes* are MN, NO; bisect MN in G and NO in g, then will G,g be their *centers of gravity.* Let $ABCD$ be suspended from the hook P, by the string PG attached to its center of gravity; and let $CDEF$ be suspended in the same manner from the hook Q,

by

by the string Qg; let them also be so placed that their ends may be *contiguous to each other*. Then will $ABCD$ balance itself about G, and $CDEF$ about g, so that the two axes NM, NO (after the cylinders are suspended) will lie in the *same straight line*; and the *pressures* upon G, g, will be the same as if the whole weights of the cylinders were col-



lected respectively in those points. Suppose now the two ends which are contiguous to each other, to be *firmly cemented together*, so that the two cylinders should become *one mass*; this will not at all effect the pressures upon G, g, but will merely serve to connect those two points together, in such a manner that the pressures upon them may be considered as acting at the extremities of an *inflexible rod* Gg *void of gravity*. Bisect the axis MO of the whole cylinder $ABFE$ in the point K, and K will be *its* center of gravity; let the prop KL be placed under K, and let the two strings PG, Qg, by which it is suspended, be removed, and the

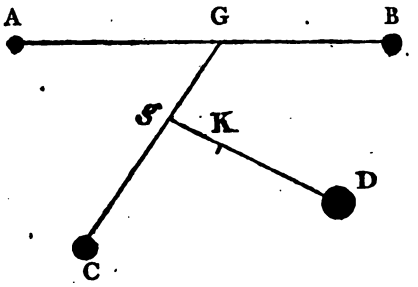
the cylinder will then balance itself about the point K; or, in other words, the two pressures acting at G, g, will be in equilibrio about that point. It only remains, therefore, to find the relation of KG to Kg; now $MK = \frac{1}{2} MO$, and $MG = \frac{1}{2} MN$, $\therefore MK - MG$ (or KG) $= \frac{1}{2} \overline{MO - MN} = \frac{1}{2} NO$; again, $OK = \frac{1}{2} MO$, and $Og = \frac{1}{2} NO$, $\therefore OK - Og$ (or Kg) $= \frac{1}{2} \overline{MO - NO} = \frac{1}{2} MN$; hence $KG : Kg :: \frac{1}{2} NO : \frac{1}{2} MN :: NO : MN ::$ cylinder $CDEF$: cylinder $ABCD$.[a] But the pressure upon G (P) is equivalent to the weight of the cylinder $ABCD$, and the pressure upon g (p) to that of the cylinder $CDEF$, $\therefore KG : Kg :: p : P$ or $P : p :: Kg : KG$, i. e. two *weights* or *pressures* P, p, acting at the extremities of the inflexible rod Gg void of gravity, will be in equilibrio about the point K, when their distances from that point are to each other *inversely* as those weights or pressures.

6. This furnishes us with the method of finding the common center of gravity of *any number of bodies whatever*, connected together by inflexible rods void of gravity.

Let

(a) For cylinders of the *same diameter* are to each other as their *axes*.

Let A, B, C, D, &c. be the bodies, and let the centers of gravity of A and B be connected together by the inflexible line AB;



take, $A : B :: BG : AG$,(a)

or, comp^{do}, $A+B : B :: BG+AG\,(AB) : AG$, then will the bodies A and B balance themselves about G, and consequently G will be their *common center of gravity* (by Art. 5).

Next, let the center of gravity of C be connected with G by the inflexible line CG, then, for the reason assigned in Art 3, the pressure upon G will be the same as if a body equal to $A+B$ was placed at G; take, therefore,

$$A+B : C :: Cg : Gg,$$

or, comp^{do}, $A+B+C : C :: Cg+Gg\,(CG) : Gg$, then g will be the center of gravity of $A+B$ and C,

(a) Since $A : B :: BG : AG$, by multiplying extremes and means $A \times AG = B \times BG$; i. e. when two bodies are in equilibrio, *the product of each body into its distance from the center of gravity is equal on each side that center.* These quantities, $A \times AG$, and $B \times BG$, therefore express the respective forces by which A and B counteract each other's effects in their tendency to motion round G.

C, and consequently the *common center of gravity* of the three bodies A, B, C.

Again, let the center of gravity of D be connected with g by the inflexible line Dg, then the pressure upon g will be the same as if $A+B$ and C (i. e. $A+B+C$) were placed at g. Take, therefore,

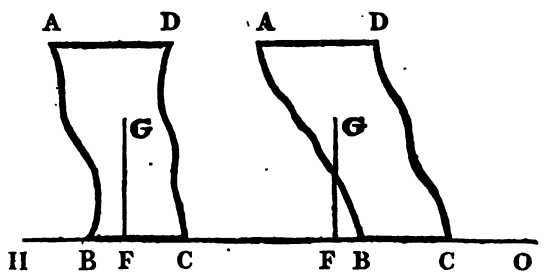
$$A+B+C : D :: DK : gK,$$

or, comp^{do}, $A+B+C+D : D :: DK+gK\,(Dg) : gK$, then K will be the center of gravity of $A+B+C$ and D, and consequently the *common center of gravity* of the four bodies A, B, C, D; moreover, the *pressure* upon K will be the same as if $A+B+C+D$ were placed at K; and thus we might proceed for *any number* of bodies.

7. It is evident that the foregoing demonstration does not all depend upon the *number* or *weight* of the bodies, or their *distance* from each other; it rests merely on the supposition that their centers of gravity are connected together by inflexible rods void of gravity. It may therefore be applied to any number of particles of matter situated in *different planes*, and placed at all possible distances from each other. Increase the number of these particles till they become *contiguous* to each other, and for the *imaginary* line void of gravity substitute the *power of cohesion*, then the system of bodies A, B, C, D, &c. may represent an irregular mass of compact matter, not unlike such as are to be met with in the works of

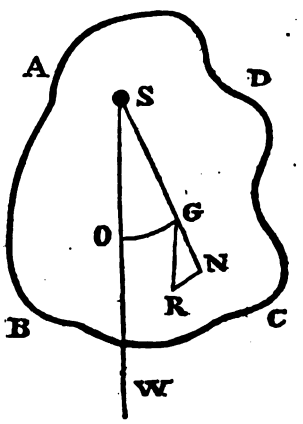
of nature or of art; and although it may be difficult to find the actual center of gravity of such a mass, yet the *latter part* of our rule still remains true,; viz. "*that if this mass be supported, its pressure downwards will be the same as if the whole quantity of matter contained in it was concentrated in its center of gravity.*"

8. From hence it follows, that whatever be the *form* or *dimensions* of a body placed upon a plane *parallel* to the horizon, it will remain *at rest*, if



the line drawn from its center of gravity *perpendicular* to the horizon falls *within its base.* For let $ABCD$ represent the section of a body passing through its center of gravity of G, and draw GF perpendicular to HO the plane upon which it stands; then, since the tendency of the body to descend is the same as if its whole weight was concentrated in G, it will *rest* or *fall* according as G is *supported* or *not*, i.e. according as F falls *within* or *without* the base BC; moreover, the *stability* of the body will depend upon the distance at which the point

point F falls *within* the base. From this property of the center of gravity it also follows, that if a body be *suspended* from any point, it will not rest *till the line which joins the center of gravity and point of suspension is perpendicular to the horizon.* For let $ABCD$ represent the *section* of a body as before, G *its center of gravity*, S *the point of suspension*; join SG, and draw SOW perpendicular to the horizon; produce SG to N, and draw GR parallel to SW; then, since the weight of the body may be considered as collected in G, its *tendency to motion* will be along the line GR. Let GR therefore represent this tendency, which resolve into GN in the direction SG, and RN perpendicular to it; the part GN is counteracted by the reaction from the point of suspension S, and NR is employed in producing motion in the direction of the circular arc GO; G therefore (and consequently the *body*) will not remain at rest till NR vanishes, i. e. till the angle NGR $(=OSG)$ vanishes, or SG *coincides with* SO. We proceed to apply the principles just now investigated, to the solution of a few practical examples.

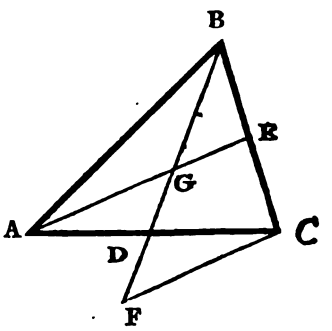


EXAM.

EXAMPLE 1.

To find the center of gravity of the triangle ABC.[a]

Bisect the side AC in D, and join BD, which will bisect all lines drawn *parallel to* AC; consequently, if BD be supported, the parts ABD, DBC of the triangle ABC will balance themselves on each side of it; hence the center of gravity is in the line BD. Bisect the side BC in E, and join AE; then, for the same reason as before, the center of gravity will be somewhere in the line AE; it must therefore be at their common intersection G. Produce now BD to F, and draw CF parallel to EA; then since $BE = EC$, BG will be equal to GF; but in the two triangles AGD, DFC, there is *one side and two angles equal to each other*, $\therefore GD = DF$, and consequently GF (or BG) $= 2GD$; hence $BG = \frac{2}{3}BD$, and $GD = \frac{1}{3}BD$.



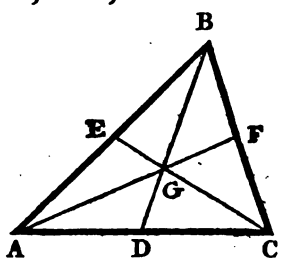
EXAM.

(a) In finding the centers of gravity of plane figures, a *lamina of matter* of an uniform density, in the shape of those figures, is of course understood.

EXAMPLE 2.

Given the three sides of the triangle ABC, to find the distance of its center of gravity from each of its angles.

Bisect the three sides AB, BC, CA in the points E, F, D, and join AF, BD, CE, which will intersect each other in the center of gravity G; then, from what was shewn in the preceding example, we have,



$BG=\frac{2}{3}BD$, $\therefore BD=\frac{3}{2}BG$, and $BD^2=\frac{9}{4}BG^2$.

$CG=\frac{2}{3}CE$, $\therefore CE=\frac{3}{2}CG$, and $CE^2=\frac{9}{4}CG^2$.

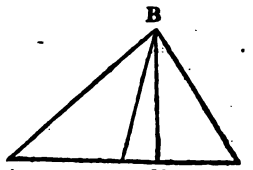
$AG=\frac{2}{3}AF$, $\therefore AF=\frac{3}{2}AG$, and $AF^2=\frac{9}{4}AG^2$.

Now let $AB=a$, $BC=b$, $cA=c$; then $AE^2=\frac{1}{4}a^2$, $CF^2=\frac{1}{4}b^2$, and $AD^2=\frac{1}{4}c^2$; but

(b) $AB^2+BC^2=2AD^2+2BD^2$, or $a^2+b^2=\frac{1}{2}c^2+\frac{1}{2}BG^2$ (A).
$AC^2+AB^2=2CF^2+2AF^2$, or $c^2+a^2=\frac{1}{2}b^2+\frac{1}{2}AG^2$ (B).
$AC^2+BC^2=2AE^2+2CE^2$, or $c^2+b^2=\frac{1}{2}a^2+\frac{1}{2}CG^2$ (C).

From

(b) For let ABC be any triangle; bisect AC in D, and draw BN at right angles to it; then,



By Euc. 12. 2,
$AB^2=AD^2+BD^2+2AD\times DN$,
and Euc. 13. 2,
$BC^2=AD^2+BD^2-2AD\times DN$
(for $CD=AD$), $\therefore AB^2+BC^2=2AD^2+2BD^2$; and by bisecting the other sides of the triangle, we obtain equations (B) and (C).

From equation (A) we have $BG=\frac{1}{3}\sqrt{2a^2+2b^2-c^2}$.

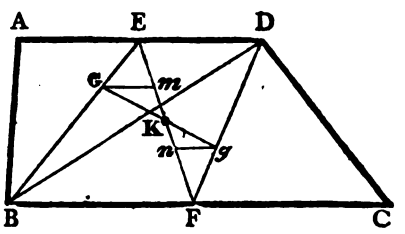
(B) $AG=\frac{1}{3}\sqrt{2c^2+2a^2-b^2}$.

(C) $CG=\frac{1}{3}\sqrt{2c^2+2b^2-a^2}$.[a]

EXAM. 3.

To find the center of gravity of a trapezium $ABCD$, whose two sides AD, BC are parallel.

Bisect AD, BC, in E, F, and join EF; then since GF bisects AD, BC, it will bisect all lines drawn parallel to BC[b], and, consequently, the center of gravity of the trapezium is in the line EF. Join BE, BD, DF, and take $GE=\frac{1}{3}BE$, $gF=\frac{1}{3}DF$; then G is the center of gravity of the triangle ABD (Ex. 1.)



and

(a) By way of corollary to this Example, it may be observed, that if a body be placed at G, and acted upon by three forces represented in quantity and direction by GA, GC, GB, it will remain *at rest*. For the force compounded of GA, GC would be the diagonal of a parallelogram whose sides are GA, GC, which (since the diagonals of parallelograms *bisect* each other) is equal to $2GD$, i. e. to GB. Hence a force represented in quantity and direction by GB would counteract the effects of GA, GC, and therefore a body thus placed would remain at rest. (Fig. page 103.)

(b) For if BA, FE, CD be produced, they will meet in the same point, which will be the *vertex* of a triangle whose *base* is BC;

and g the center of gravity of the triangle BDC. Join Gg; then, conceiving the triangles ABD, BDC to be collected in G, g, their *common center of gravity* must be in the line Gg; i. e. the center of gravity of the *trapezium* $ABCD$ must be in the line Gg; it is *also* in the line EF; consequently it is in K, the intersection of EF and Gg. Draw Gm, gn parallel to AD or BC; then since $EG = \frac{1}{3}BE$, Em must be equal to $\frac{1}{3}EF$; and for the same reason $Fn = \frac{1}{3}EF$; $\therefore Em = mn = nF$. Now K being the common center of gravity of the triangles ABD, BDC, we have

$$GK : Kg :: \Delta BDC : \Delta ABD :: BC : AD^{(b)},$$

but by sim. Δ's $GK : Kg :: Km : Kn$, $\therefore Km : Kn :: BC : AD$.

Hence we have, 1st,

$Km : Km + Kn\ (mn = Em) ::$ BC $: BC + AD$,

& $Km + Em\ (EK) : mn ::$ $2BC + AD : BC + AD\ (A)$.

2d, $Km + Kn\ (mn = nF) : Kn ::$ $BC + AD : AD$,

& $mn : Kn + nF\ (FK) ::$ $BC + AD : BC + 2AD\ (B)$.

ex æquo (A) and (B),

then $EK : FK ::$ $2BC + AD : BC + 2AD$.

If, therefore, the *parallel* sides of the trapezium be bisected, and if the line which joins the points of

BC; and since EF bisects the base BC, it will bisect all lines drawn *parallel* to it.

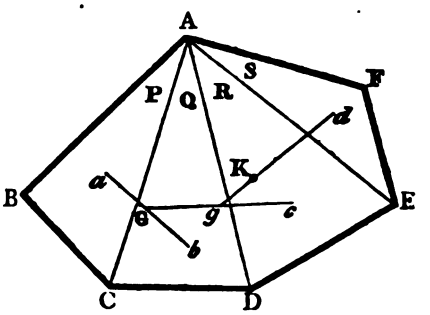
(b). For the triangles ABD, BDC, are of the *same altitude*; $\therefore$ (Euc. 1. 6.) $ABD : BDC :: AD : BC$.

of bisection be divided in the ratio of $2BC+AD : BC+2AD$, it will determine the center of gravity.

EXAM. 4.

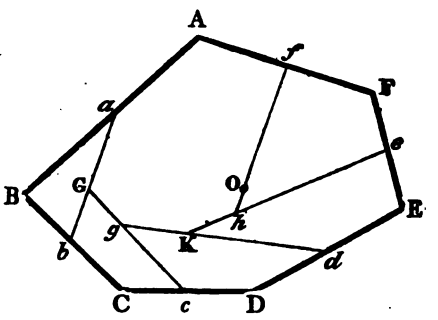
To find the center of gravity of an irregular polygon.

Every polygon whose sides and angles are given may be divided into triangles whose *areas* and *centers of gravity* may be found by the rules of trigonometry. Let $ABCDEF$ be an irregular polygon, divided into triangles whose *areas* are represented by P, Q, R, &c. and whose *centers of gravity* are respectively a, b, c, &c. Conceive these triangles to be collected in the points a, b, c, &c.; join ab, and take $bG : aG :: P : Q$, then G will be the center of gravity of the figure $ABCD$. Join Gc, and take $cg : Gg : P+Q : R$; then g will be the center of gravity of the figure $ABCDE$. Let g and d be joined, and make $dK : gK :: P+Q+R : S$, then K will be the center of gravity of the whole polygon; and so we might proceed, whatever be the number of sides.



If it was required to find the center of gravity of

of the *perimeter* of the polygon; then bisect the sides in the points *a*, *b*, *c*, &c. and (since the center of gravity of a right line is evidently in its *middle point*) *a*, *b*, *c* &c. will be the centers of gravity of the sides *AB*, *BC*, *CD*, &c. respectively. Join *ab*, and take $bG : aG :: AB : BC$; then *G* would be the center of gravity of that part of the perimeter represented by *ABC*. Again, join *Gc*, and take $cg : Gg :: AB + BC : CD$, then *g* is the center of gravity of such part of the perimeter as is represented by *ABCD*; and so we might proceed till we had found (*O*) the center of gravity of the whole perimeter.

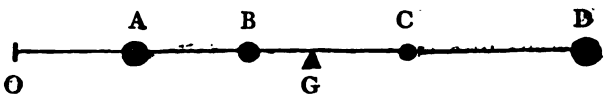


EXAM. 5.

To find the common center of gravity of any number of bodies placed in a right line. (Fig. in p. 108.)

Let the bodies *A*, *B*, *C*, *D* be so placed, that the line *OD* may pass through their respective centers of gravity; it is required to find the distance of their common center of gravity from the point *O*. Suppose *OD* to be an inflexible line void of gravity, and let *G* be the common center of gravity of the bodies; then if *G* be supported, the effort of each body

body to produce motion round G would be measured by *the product of its weight into its distance from G;*



(See Note, page 97), *i. e.* the effort of $A = A \times AG$; of $B = B \times BG$; &c. &c; and as the bodies are supposed to be in *equilibrio* about G, the *sum of their efforts* on each side G must be *equal* to each other, or

$$A \times AG + B \times BG = C \times CG + D \times DG, \text{ i.e. } A \times \overline{OG - OA} + B \times \overline{OG - OB} = C \times \overline{OC - OG} + D \times \overline{OD - OG}; \therefore$$

$$A \times OG + B \times OG + C \times OG + D \times OG = A \times OA + B \times OB + C \times OC + D \times OD;^{(a)}$$

$$\text{hence } OG = \frac{A \times OA + B \times OB + C \times OC + D \times OD}{A + B + C + D}.$$

From which we infer, that the distance of the common center of gravity of *any number* of bodies thus placed is equal to *the sum of the products arising from multiplying each body into its distance from some given point divided by the sum of the bodies.*

Cor. If $OA = 0$, then $AG = \frac{B \times AB + C \times AC + D \times AD}{A + B + C + D}$;

and

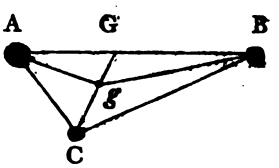
(a) By actual multiplication and transposition.

and if the bodies lie on *different sides* of the point O, (for instance, A, B on *one side*, and C, D on the *other*,) then $OG = \frac{A \times OA + B \times OB - C \times OC - D \times OD}{A + B + C + D}$.

EXAM. 6.

Three bodies A, B, C, weighing respectively 3, 2, *and* 1 *pounds, have their centers of gravity joined by the lines AB, BC, CA; of which $AB = 5$ feet, $BC = 4$, $CA = 2$; What is the distance of their common center of gravity from the body C?*

Since the three sides of the triangle ABC are 5, 4, 2, the three angles A, B, C will be found, by the rules of Trigonometry, to be respectively 49°. 27′, 22°. 19′, and 108°. 14′. Let G be the center of gravity of the bodies A and B, and g the common center of gravity of the three bodies found as in Art. 6; then since $AB = 5$, $A = 3$, $B = 2$, and $A : B :: BG : AG$, AG will be equal to 2 feet, and BG to three feet; hence in the triangle GAC there is given $AC = 2$, $AG = 2$, and the angle $CAG = 49°. 27′$, $\therefore$ each of the angles AGC, $ACG = 65°. 16\frac{1}{2}′$, from which CG is found to be equal to 1.673. But from the rule, $Cg : Gg :: A + B : C$, $\therefore Cg : Cg + Gg \ (= CG = 1.673) :: A + B \ (5) : A + B + C \ (6)$; hence $Cg = \frac{1.773 \times 5}{6} = 1.394$ feet.



Ex.

EXAM. 7.

Given the distance of three bodies A, B, C, from their common center of gravity g, *and the angles AgB, BgC, AgC; to compare the weights of the bodies.*

Join Ag, Bg (Fig. in Ex. 6.) and let the sines of the angles AgB, BgC, AgC, be respectively a, b, c; produce Cg to G, and G will be the center of gravity of A and B;[a] for the same reason, if Ag was produced, it would pass through the center of gravity of B and C; and if Bg was produced, it would pass through that of A and C. Let sin. $\angle AGg$ or $BGg = s$. Now

$$\text{sin.}\angle AGg\ (s) : \text{sin.}\angle AgG\ (=\text{sin. } AgC=c) :: Ag : AG=Ag\times\frac{c}{s},$$

and

$$\text{sin.}\angle BGg\ (s) : \text{sin.}\angle BgG\ (=\text{sin. } BgC=b) :: Bg : BG=Bg\times\frac{b}{s}.$$

Hence, $A : B\ (:: BG : AG) :: Bg\times\frac{b}{s} : Ag\times\frac{c}{s} :: Bg\times b : Ag\times c.$

For the same reason,

$$B : C :: Cg\times a : Bg\times c,$$

$$\therefore \text{ ex aequo, } A : C :: Cg\times a : Ag\times b.$$

Cor.

(a) For it is evident, that if Cg be produced, it must meet AB in such a point, that $A+B$, when placed at G, must balance C; the point G is therefore the center of gravity of A and B. In the same manner, if Ag, Bg were produced, they would meet BC, AC, in such points, that $B+C$ must balance A, and $A+C$ must balance B, which points are the common centers of gravity of *B and C*, and of *A and C* respectively.

Cor. If $a=b=c$; i. e. if the angles at g are each of them 120°, then $A : B :: Bg : Ag$; $B : C :: Cg : Bg$; and $A : C :: Cg : Ag$.

EXAM. 8.

A cylindrical tower, consisting of uniform materials closely cemented together, is 20 feet high; and the diameter of its base is 4 feet; How far may it deviate from its perpendicular position, before it is in danger of falling.

Let $ABCD$ represent a section of the tower passing through its axis EF, and let G be its center of gravity. Suppose it to be so much inclined, that the perpendicular line GB, let fall from G, falls upon the edge of its base BC; then GF (10) : BF (2) :: radius (1) : cos. $\angle GFB$;



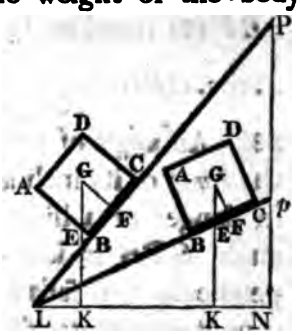
$\therefore$ cos. $GFB = \frac{2}{10} = .200 =$ cos. $\angle 78°$ 33′. An angle of 78°. 33′ is therefore the *limit* of its inclination, before it is in danger of falling (Art. 8). If the angle GFB is less then 78°. 33′, then the perpendicular GB falls *without* the base, and the tower cannot sustain itself.

EXAM. 9.

A piece of timber of uniform density and prismatic form, a section of which (perpendicular to its sides, and

and passing through its center of gravity (G) is represented by the square ABCD, is placed upon an inclined plane; it is required to shew when it will have a tendency to roll, *and when to* slide *down the plane.*

Draw *GF* perpendicular to the plane, and *GK* perpendicular to its *base*, and let *PLN* be *greater* and *pLN less* than half a right angle. In the *former* case, since *ELK* is greater than 45°, *LEK* or *GEF* will be less than 45°; ∴ the angle *EGF* is greater then the angle *GEF*, and consequently *EF* is *greater* than *GF* or *BF*; hence the body has always a tendency to *fall over* in the direction *GE*, and will therefore *roll* down the plane *PL*. In the *latter* case the angle *EGF* is *less* than *GEF*, ∴ *EF* is *less* than *GF* or *BF*; the whole weight of the body therefore presses upon the plane *pL*. Let *GE* represent this weight, which resolve into two, *GF*, *FE*; *GF* will represent the reaction of the plane upon the body, and *FE* will represent a force which tends to make the body *slide* down the plane. Hence it appears, that the body will have a tendency either to *roll* or *slide*, according as the angle of the plane's inclination is *greater* or *less* than 45°.



XV.

XV.

In considering the circumstances under which a body would *slide* or *roll* down an inclined plane, it should be observed, that if the surfaces of the body and the plane be *perfectly smooth*, no *rolling* will take place, whatever be the angle of inclination of the plane. To give a body a tendency to *rotatory* motion about its center of gravity (G), it is evident that there must be some mutual action between the surface of the body and the surface of the plane (such as that, for instance, which arises from *friction*, the *unrolling* of rope, &c. &c.); if there be not some such action as this, all the parts of the body being *equally* accelerated, the body will, under all circumstances, *slide* down the plane. (See Note *a*, page 58, vol. II.)

Questions for Practice.

1. If three *equal* bodies be placed at the angles of any triangle; shew that the common center of gravity of those bodies is in the same point with the center of gravity of the triangle.

2. Four bodies A, B, C, D, weighing respectively 1, 2, 4, and 7 pounds, are placed with their centers of gravity in a right line, at the distance of 3, 5, 7,

and 9 feet from a given point; What is the distance of their *common* center of gravity from that given point; and between which two of the bodies does it lie?

Ans. Between C and D; and its distance from the given point $7\frac{3}{7}$ feet.

3. The bodies A, B, C, weighing respectively 5, 3, and 12 pounds, are so placed, that $AB = 8$ feet, $AC = 4$ feet, and the angle BAC a *right angle*; What is the distance of their common center of gravity from the body C? Ans. 2 feet.

4. Three bodies A, B, C, are so situated with respect to their common center of gravity (g), that $Ag = 10$, $Bg = 15$, $Cg = 20$ feet; the angles AgB, BgC, and AgC, are each of them 120°; A weighs 12 pounds; What are the weights of B and C?

Ans. $B = 8$lbs. $C = 6$lbs.

5. Supposing the height of the cylinder in Exam. 7. to be only *twice the diameter of its base*; what is the limit of its angle of inclination before it is in danger of falling? Ans. 60°.

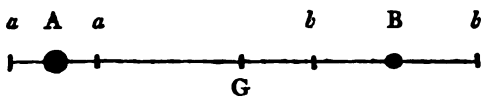
We have thus shewn the method of finding the center of gravity of a body or system of bodies in a few ordinary cases; and have at the same time established this fundamental principle, that, *when the body or system is supported, its pressure (or tendency*

tendency to motion) is the same as if all the matter contained in the body or system was collected in that center of gravity. We are next to consider how the common center of gravity of a system of bodies is affected, when some or all of them are *actually in motion.*

XVI.

On the Method of investigating the Effect produced upon the Common Center of Gravity of a System of Bodies, when some or all of them are actually in motion.

1. Let A and B be two unequal bodies; then, if they approach to or recede from each other with



velocities inversely proportional to their weights, their common center of gravity G will remain *at rest.* For take $Aa : Bb :: B : A$, and suppose A to move through Aa whilst B moves through Bb, then (since $V \propto S$ when T is given) *velocity of* A : *velocity of* B :: ($Aa : Bb$::) $B : A$; hence we have

$$AG : BG :: B : A,$$

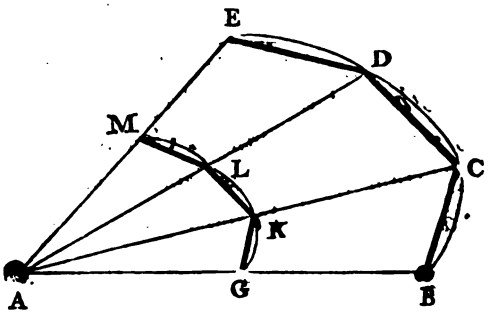
and $$Aa : Bb :: B : A;$$

$$\therefore AG \pm Aa : BG \pm Bb :: B : A, \text{ i.e. } aG : bG :: B : A;$$

from which it appears that G is their common center of gravity when the bodies are arrived at a and

a and b, i. e. the center of gravity has remained *at rest* whilst the bodies have approached to or receded from each other through the spaces Aa, Bb.

2. Suppose now that one body (A) remains *at rest*, whilst the other body (B) moves uniformly along the sides BC, CD, DE, of the *polygon*

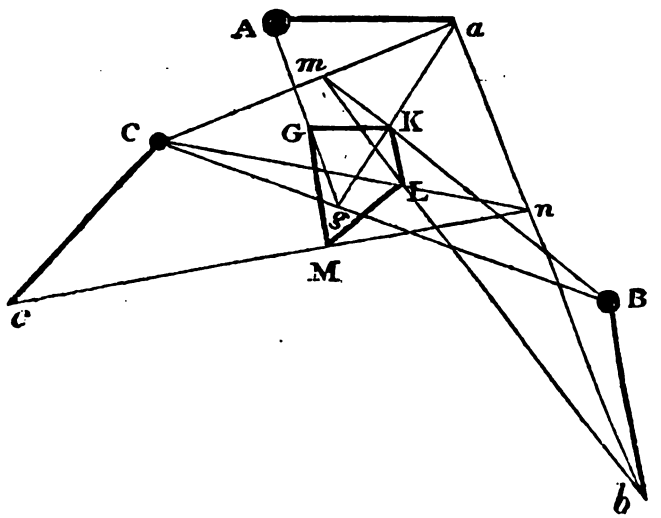


$BCDE$. When the body arrives at C, join AC, and take $AK : KC :: B : A$, or $AK : AC :: B : A+B$, then K will be the place of the center of gravity. When the body arrives at D, E, join also AD, AE, and divide them in the points L, M in the ratio of $B : A$, then will L, M be the position of the center of gravity at the end of those respective times. Let GK, KL, LM be now joined; and since $AG : AB :: B : A+B$, and $AK : AC :: B : A+B$, $\therefore AK : AC :: AG : AB$; hence GK is parallel to BC, and the triangle AGK similar to the triangle ABC. In the same manner it may be proved that the triangles AKL, ALM are respectively similar to ACD, ADE, and the whole figure $AGKLM$ to the

the polygon $ABCDE$. Whilst the body B therefore moves uniformly along the sides of the polygon $BCDE$, the common center of gravity G describes with a uniform motion a *similar* polygon $GKLM$; and since, from the nature of similar figures, $GK + KL + LM : BC + CD + DE$:: ($AG : AB$, i. e.) $B : A + B$, the *velocity of the center of gravity* will be to the *velocity of the body* B :: $B : A + B$. Suppose now the number of the sides of the polygon $BCDE$ to be increased without limit, so that it may be considered as assuming the form of a *Curve*, then shall we come to this *general conclusion*, that, *whilst the body B proceeds uniformly along the perimeter of the figure BCDE (whether rectilinear or curvilinear), the center of gravity G will describe with a uniform motion a similar figure GKLM, with a velocity which is to that of B as* $B : A + B$.

3. Let us next take the case of three bodies A, B, C (Fig. in p. 118.) moving with uniform velocities, in *equal successive parts of time*, through the spaces Aa, Bb, Cc. Let G be the position of the common center of gravity of *the three bodies*, and g that of B and C, *before they begin to move*; then (by Art. 6. Sect. XIV.) $Gg : Ag :: A : A + B + C$. Whilst A moves from A to a, $B + C$ may be considered as *at rest in* g, $\therefore$ (by Art. 2.) the common center of gravity (G) will in the same time describe GK parallel to Aa, and GK will be to Aa :: ($Gg : Ag$::) $A : A + B + C$. When A is arrived

arrived at a, join aC and BK; produce BK till it meets aC in m, then (see Note, page 110.) m will be



the center of gravity of A *and* C; join mb, then whilst B moves from B to b, the common center of gravity will describe KL parallel to Bb, and KL will be to Bb :: (mK : mB ::) B : $A+B+C$. When B is arrived at b, join ab, CL; produce CL till it meets ab in n, then n will be the center of gravity of A *and* B; join nc, then whilst C moves from C to c, the common center of gravity will describe LM parallel to Cc, and LM will be to Cc :: (nL : nC ::) C : $A+B+C$. Whilst the bodies A, B, C, therefore, in equal successive parts of time, move uniformly through the spaces Aa, Bb, Cc, their *common center of gravity* will in the same time *describe the polygon* $GKLM$, whose sides GK,

GK, KL, LM are respectively *parallel* to *Aa, Bb, Cc*, and bear to them the ratio of *A, B*, and *C* : $A+B+C$.

4. If, instead of moving in *successive* intervals of time, the three bodies *A, B, C* were all to begin to move *at the same instant*, and describe the lines *Aa, Bb, Cc, cotemporaneously*, let us then consider what effect would be produced upon their common center of gravity. Now since *GK, Aa* are described in the *same time*, the velocity of the common center of gravity (v) : velocity of the body A (V) :: GK : Aa :: A : $A+B+C$; hence $A \times V = \overline{A+B+C} \times v$; i. e. the *momentum of* A[a] is equal to the *momentum of a body* $= A+B+C$ moving with the velocity (v); the *same force* therefore which causes the body A to move over Aa, would in the *same time* cause a body equal to $A+B+C$ to move over GK.[b] For the same reason, the forces which impel B and C over Bb, Cc, are such as would in the *same time* cause a body $= A+B+C$ to move over

(a) By Sect. II. $A \times V$ represents the *momentum* of A moving with the velocity (V), and $\overline{A+B+C} \times v$ represents the momentum of a body equal to $A+B+C$ moving with the velocity (v); where the letters A, B, C in course represent the weights or *quantities of matter* in the bodies A, B, C.

(b) For the *forces* are proportional to the *momenta generated in a given time.*

over KL, LM. Hence it appears, that the motion of the common center of gravity along the sides of the polygon $GKLM$ is analogous to the motion of a body equal $A+B+C$ acted upon by 3 forces which would carry it over GK, KL, LM in the same time that they would carry the bodies A, B, C over the spaces Aa, Bb, Cc respectively. But a body acted upon *at once* by these forces would (by Art. 5. Sect. VIII.) describe the other side GM of the polygon $GKLM$ in the same time that it would describe either of the sides GK, KL, LM, when the forces act *separately*; if the bodies A, B, C, therefore, move *cotemporaneously*, their common center of gravity will describe the line GM, whilst the bodies themselves describe the three lines Aa, Bb, Cc.(a)

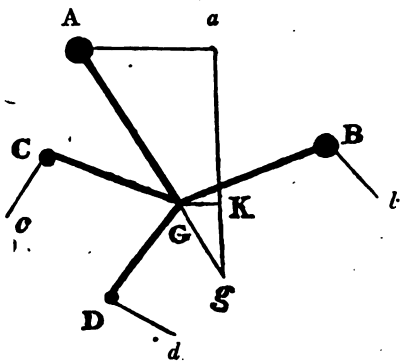
5. It is evident, that the reasoning contained in the last article may be extended to *any number* of bodies; we are therefore arrived at this general conclusion, that, *when a system of bodies are in motion, their*

(a) We have here supposed the bodies A, B, C to have their centers of gravity in *the same plane*; in which case it is evident that the motion of their common center of gravity will be in the same plane. If the motion of the bodies be in *different planes*, then the value of each line GK, KL, LM might be found as before; but as they will then lie in different planes, the resulting quantity GM must be ascertained according to the principles laid down in Sect. XII. Lect. 3.

their common center of gravity will move in the same manner as if a body equal to the sum of the bodies was placed in that point, and the same motions were communicated to it as are communicated to the bodies separately. From which it appears in the *first place*, that if the bodies which compose the system *move uniformly in right lines*, then their common center of gravity will either remain *at rest*, or will move *uniformly in a right line;* for if a body equal to the sum of the bodies were placed in that center, and then acted upon by the same forces which cause the bodies to move separately in right lines, it would either remain *at rest* (viz. *when the forces counteract each other*), or would describe uniformly the *remaining side* of a polygon whose *other sides* represent the quantity and direction of the several forces acting upon it. In the *second* place, it may be observed, that the common center of gravity of the system will not be affected by the *mutual action of the bodies upon each other;* for action and re-action being equal, the effect produced upon the common center of gravity by such mutual action will only be that of two equal and opposite forces acting upon a body equal to the sum of the bodies placed in that center; which would evidently not disturb its state, either of *motion* or *quiescence*. *Lastly*, it appears, that if the motion of the bodies in these right lines was to cease, and they were left to the mutual *attraction* of each other, then their common

center of gravity would remain *at rest*, and the bodies would *approach* each other in lines drawn to it from their *respective* centers of gravity, and all collect together in that common center.[a]

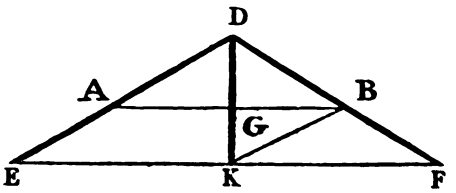
(a) This will probably require a little explanation. What we mean is this. If the motion of the bodies *A*, *B*, *C*, &c. in the directions *Aa*, *Bb*, *Cc*, &c. were to *cease*, and they were left to the mutual *attraction* of each other, their common center of gravity (*G*), being under the same circumstances as a body (equal to the sum of the bodies) acted upon by equal and contrary forces, would remain *at rest*; but at the same time it is evident, that the bodies would *move towards each other*, and this motion we assert would be along the lines *AG*, *BG*, *CG*, &c. which meet in *G*. For suppose any one of them (*A* for instance) to move out of the line drawn from *A* to *G*, in the direction *Aa*; then if *g* be the common center of gravity of the *remaining* bodies, (by Art. 3.) whilst *A* moves through *Aa*, *G* will describe the line *GK* parallel to *Aa*; and the same might be proved of the other bodies *B*, *C*, *D*, &c. In this case, therefore, the common center of gravity would *not* be at rest, which is contrary to the principle just now explained. Hence the bodies *A*, *B*, *C*, &c. in their approach towards each other, cannot possibly *deviate* from the lines *AG*, *BG*, *CG*, &c.; they will consequently all collect together at the point *G*.



EXAM. 1.

Let two equal bodies, A and B, move from the point D, with the same uniform velocity, along the sides DE, DF of the isosceles triangle DEF, whose angle EDF = 120°; It is required to compare the velocity of their common center of gravity with that of either of the bodies A or B.

When the bodies are arrived at the points *A*, *B*, join *AB*; and since the bodies move with the same uniform velocity, $DA = DB$; $\therefore DA : DB :: DE : DF$, and AB is parallel to EF.



Again, because $A = B$, the center of gravity G will bisect AB; hence, whilst the bodies move uniformly along DE, DF, the center of gravity will move through the line DK, which *bisects* (and is also at *right angles* to) AB, EF. Now the $\angle EDF = 120°$, $\therefore EDK = 60°$, and $DEK = 30°$; but since DE, DK are described in the same time, *velocity of A : velocity of center of gravity* :: $DE : DK$:: rad. : sin. 30° :: 2 : 1; the center of gravity therefore moves with *half* the velocity of either of the bodies *A* or *B*.

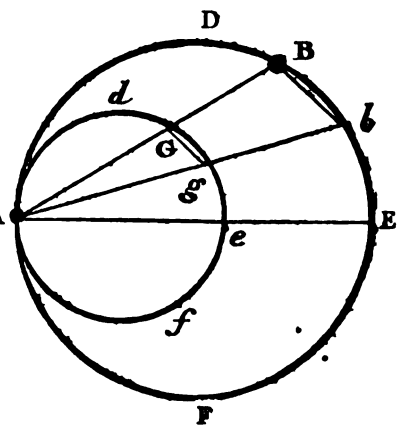
The same conclusion may be deduced from the principles laid down in Art. 4. of this Section; for suppose the two bodies to be placed at *D*, and take $DB = BF$; then if *A* remains *at rest* whilst *B* moves uniformly

uniformly along DF, the center of gravity will in that time move from D to B; and if B remains at rest in F whilst A moves from D to E, the center of gravity will describe the line BK parallel to DE; and since the side DF is bisected in B, the side EF will be bisected in K. Suppose now the bodies A and B to move *cotemporaneously*, then their common center of gravity will describe DK, the third side of the triangle DBK, in the same time that it would have described either of the other two sides (DB, BK) when the bodies move *successively*.

EXAM. 2.

Let the two bodies A and B be placed at the extremity A of the diameter of the circle ADF; and then let B describe the circle ADF whilst A remains at rest in the point A; In what manner will their common center of gravity move?

Let B, b be any two positions of the body B, and G, g, the corresponding positions of the common center of gravity of A and B. Join Bb, Gg; then, by Art. 2, Gg is parallel to Bb, $\therefore$ the $\angle AGg = \angle ABb$, hence ADb, Adg, are *similar segments of circles;* and when B



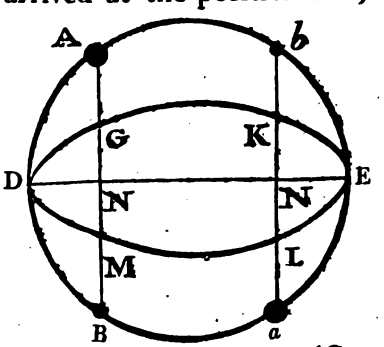
has

has described the *semicircle ADE*, the center of gravity will have described the *semicircle Ade*; and so for the semicircles on the *other side* of *AE*. Now $Ae : eE :: B : A$, $\therefore Ae : AE :: B : A+B$; hence while B describes the circle *ADF*, the common center of gravity of A and B will describe the circle *Adf*, whose diameter : diameter of $ADF :: B : A+B$. This conclusion indeed follows *immediately* from the reasoning in Art. 2; for it was there shewn that the *whole figure* described by the common center of gravity, is *similar* to that which the *moving* body (B) describes.

EXAM. 3.

Two bodies, A and B, begin to move at the same instant from the extremity D of the diameter (DE) of the circle DAEB, and continue to move on with the same uniform velocity till they meet in E; they pass each other at E, and then continue to move on till they arrive at the point D, from whence they set off; What is the course of the common center of gravity during this revolution of the two bodies?

Suppose the bodies arrived at the position *AB*, then since $DA = DB$, the line *AB* will be bisected by *DE* in *N*. Let *G* be the common center of gravity of *A* and *B*, then $A+B : B :: AB : AG$, $\therefore \frac{1}{2}\overline{A+B} : B :: \frac{1}{2}AB$ (or *AN*) : *AG*;



: AG; hence $AN : AG :: A+B : 2B$, and $\therefore AN : AN-AG\ (GN) :: A+B : A+B-2B\ (A-B)$; i.e. $AN : GN$ in the *given ratio* of $A+B : A-B$; consequently whilst the bodies A and B describe respectively the *semicircles* DAE, DBE, their common center of gravity describes the *semi-ellipse* $DGKE$; in the same manner it may be proved that whilst A and B describe the *semi-circles* EBD, EAD, their common center of gravity would describe a semi-ellipse $ELMD$, equal and similar to $DGKE$. Whilst the bodies A and B therefore perform their respective revolutions, their common center of gravity will describe the ellipse $DGEM$, whose *major axis* : *minor axis* :: ($AN : GN$::) $A+B : A-B$.

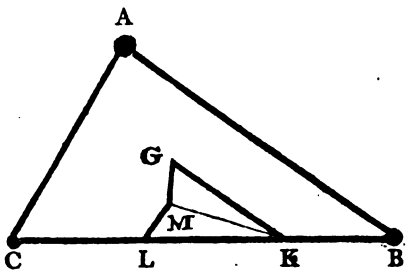
Cor. If the bodies be *equal*, $A-B=0$, and the ellipse becomes a *straight line*. Indeed it is evident that in this case the common center of gravity would move in a line which always *bisects* AB, *a b*, i.e. in the *diameter* DE.

EXAM. 4.

Three bodies, A, B, C, at the same instant begin to move uniformly from the three angles of a given triangle, and in the same time change places in the direction ABC; How will their common center of gravity be affected by this motion of the bodies?

Let G be their common center of gravity, and suppose the bodies first to move in *succession*. Whilst A moves from A to B, their common center of gravity will (by Art. 3.) describe GK parallel

parallel to AB, and GK will be to $AB :: A : A+B+C$, $\therefore GK = \frac{A \times AB}{A+B+C}$.



Whilst B moves from B to C, the center of gravity will describe $KL = \frac{B \times BC}{A+B+C}$; and if LM be drawn *parallel* to CA, and *equal* to $\frac{C \times AC}{A+B+C}$, whilst C moves from C to A the center of gravity will describe LM. Suppose now the bodies to move *cotemporaneously*, then their common center of gravity will describe GM (*the remaining side of the polygon* $GKLM$) whilst the bodies *change places* in the direction ABC.

To find the *actual value* of GM, we have KL, LM, and the $\angle KLM (= ACB)$ given, from which MK and the $\angle MKL$ may be found; but $\angle GKM = GKL$ (or ABC) $- MKL$; in the triangle GKM there are therefore given GK, KM, and the $\angle GKM$, from which GM may be determined.

Cor. If $A = B = C$, then $GK = \frac{1}{3}AB$, $KL = \frac{1}{3}BC$, and $LM = \frac{1}{3}AC$, $\therefore GK$, KL, LM are to each other as AB, BC, AC; and since the $\angle$s GKL, KLM are respectively equal to ABC, BCA, the three lines GK, KL, LM will form a triangle *similar* to the triangle ABC. GM therefore in this case is equal to 0, and the body remains at rest.

rest. This also follows from the *general theorem* in Art. 5; for the common center of gravity being under the same circumstances as a body acted upon by three forces which are to each other as the three sides of a triangle *taken in order*, will, by (Art. 1. Sect. XI.) remain *at rest*.

XVII.

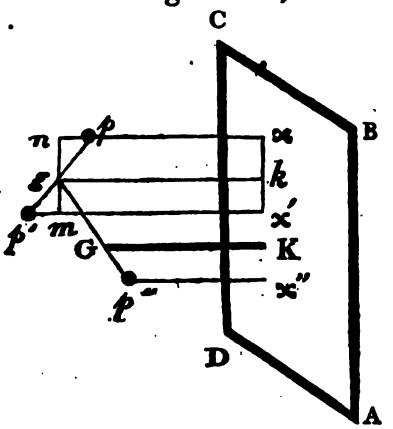
On the Investigation of General Theorems for finding the Center of Gravity of any Body or any System of Bodies whatever.

In the former part of this Lecture we shewed the method of finding the center of gravity of a body or system of bodies in a variety of particular instances. We now proceed to consider the subject in a more general point of view, by the investigation of theorems for ascertaining its position with respect to any body or system of bodies whatever.

1. Let p, p', p'', &c. be any number of small bodies or particles of matter,[a] and $ABCD$ a plane placed in any position with respect to them. Join pp', and let g be the common center of gravity of p and p'; draw px, gk, $p'x'$ at right angles to the plane $ABCD$, and consequently parallel to each other; join xx', and since the points p, g, p' are in a straight line, the points x, k, x'

(a) Or rather let them be considered as so many small bodies collected in their respective centers of gravity in the *points* p, p', p'', &c.

x, k, x' will also be in a straight line, and therefore xx' will pass through k. Join gp'', and let G be the common center of gravity of p, p', p''; draw GK, p'', x'' perpendicular to the plane; and through g draw mn parallel to xx', meeting px produced in n.



Now $p : p' :: p'g : pg$:: (by sim. Δ^s) $p'm : pn$;

$\therefore \overline{p \times pn} = \overline{p' \times p'm}$,

or $p \times \overline{nx - px} = p' \times \overline{p'x' - mx'}$; But $nx = gk = mx'$,

$\therefore p \times \overline{gk - px} = p' \times \overline{p'x' - gk}$,

and $\overline{p + p'} \times gk = p \times px + p' \times p'x'$,

$$\therefore gk = \frac{p \times px + p' \times p'x'}{p + p'};$$ for the same

reason if $p + p'$ is placed at g, we have

$$GK = \frac{\overline{p + p'} \times gk + p'' \times p''x''}{\overline{p + p' + p''}}$$

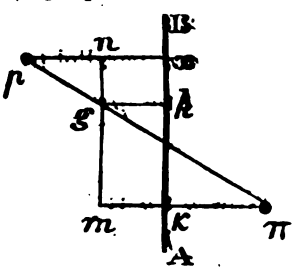
$$= \frac{p \times px + p' \times p'x' + p'' \times p''x''}{p + p' + p''};^{(b)}$$

and thus we might proceed, whatever be the number of particles. Hence "the distance of the common "center

(b) Hence $p \times px + p' \times p'x' + p'' \times p''x''$ +&c. $= \overline{p + p' + p''}$ &c. $\times GK$, which is the well-known theorem for estimating the pressure of fluids.

"center of gravity of any number of bodies or "particles of matter from a plane given in position, "is equal to *the sum of the products arising from "multiplying each body into its distance from the "plane divided by the sum of the bodies.*"

2. If there be two particles p, π on *opposite sides* of the plane, then draw px, πx perpendicular to the plane, as before, and let g be their *center of gravity*; through g draw gk *perpendicular* and mn *parallel* to the plane, and produce πx to meet mn in the point m. For the same reason as in Art. 1, we have $p \times pn = \pi \times \pi m$, or $p \times \overline{px - gk} = \pi \times \overline{\pi x + gk}$, $\therefore$ (by multiplication and transposition) $\overline{p + \pi} \times gk = p \times px - \pi \times \pi x$, and $gk = \frac{p \times px - \pi \times \pi x}{p + \pi}$. Here we may observe that the sign of πx is *negative*, and the same will be the case for *all* particles lying on the *opposite side* of AB; if therefore "p, p', p'', &c. be particles lying on *one* "side of a plane given in position, and π, π', π'', &c. "be particles lying on the *other* side, at the perpen- "dicular distances $px, p'x', p''\pi''$, &c. and $\pi x, \pi' x', \pi'' x''$, "&c. respectively; then *the distance of the common "center of gravity of* p, p', π, π', &c. *from that plane* "will be equal to

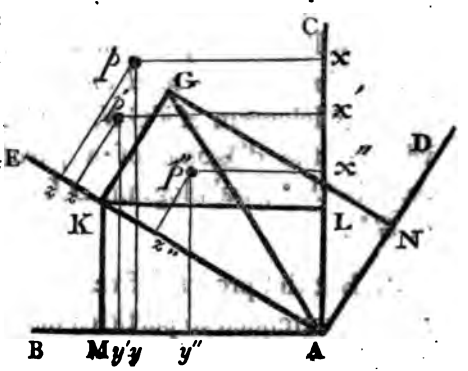


"$\frac{p \times px + p' \times p'x' + \&c. - \pi \times \pi x - \pi' \times \pi' x' - \&c.}{p + p' + \&c. + \pi + \pi' + \&c.} (= GK)$;"

where

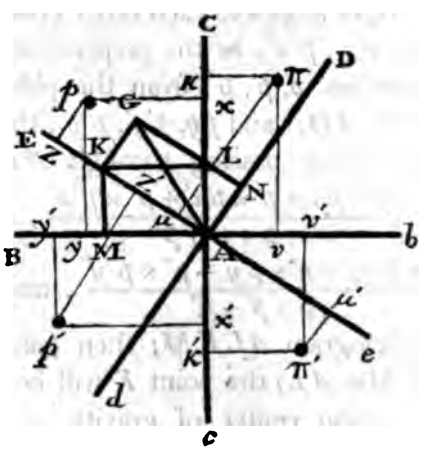
where G will lie on the *same* or *contrary* side of the plane with p, p', p'', &c. according as $p \times px + p' \times p'x' +$ &c. is *greater* or *less* than $\pi \times \pi\kappa + \pi' \times \pi'\kappa' +$ &c.

3. Let now AB, AC, AD be three straight lines issuing from the point A, at right angles to each other; in which case the *planes* passing through AB, AC; AC, AD; and AB, AD, respectively; will *also* be at right angles to each other (Euc. 11. Def. 4). Let $px, p'x', p''x''$, be the perpendicular distances of the particles p, p', p'', from the plane passing through AC, AD; and $py, p'y', p''y''$, their distances from the plane passing through AB, AD; take $AM = \frac{p \times px + p' \times p'x' + p'' \times p''x''}{p + p' + p''}$, and $AL = \frac{p \times py + p' \times p'y' + p'' \times p''y''}{p + p' + p''}$, and complete the parallelogram $ALKM$; then (since $KL = AM$, and $KM = AL$) the point K will be the position of the common center of gravity of the particles with respect to the planes passing through AC, AD; AB, AD respectively. Join AK, and produce it to E, and let fall the perpendiculars $pz, p'z', p''z''$ upon



upon AE (i. e. upon *the plane passing through* AB, AC); draw GK at right angles to AK, and make it equal to $\frac{p \times pz + p' \times p'z' + p'' \times p''z''}{p + p' + p''}$, then the point G will be the *actual* position of the common center of gravity of the particles p, p', p''; for it will be its position as referred to *the three planes forming the solid angle* $BACD$. Complete the parallelogram $AKGN$, and join AG; then AG will be the diagonal of a *parallelopipedon* whose base is the parallelogram $ALKM$ and height AN or GK.

4. But the process of reasoning in the last article may be extended to particles disposed in *any manner* round the point A. For let the lines BA, EA, CA, DA be produced to the points b, e, c, d, and at the same time conceive *the planes passing through them* to be produced; then all the particles p, p', π, π', &c. that can possibly be spread round A will lie on one side or other of those planes. Now since the particles p, p' lie on *one side* of the plane passing



passing through Cc, Dd, and the particles π, π' on the *other*, the *distance of the common center of gravity* of p, p', π, π', &c. *from that plane*, will (by Art. 2.) be equal to

$$\frac{p \times px + p' \times p'x' - \pi \times \pi x - \pi' \times \pi' x'}{p + p' + \pi + \pi'} \quad (= X.)$$

For the same reason, its distance from *the plane passing through* Bb, Dd will be

$$\frac{p \times py + \pi \times \pi\nu - p' \times p'y' - \pi' \times \pi'\nu'}{p + p' + \pi + \pi'} \quad (= Y.)$$

And its distance from *the plane passing through* Bb, Cc will be

$$\frac{p \times pz + \pi \times \pi\mu - p' \times p'z' - \pi' \times \pi'\mu'}{p + p' + \pi + \pi'} \quad (= Z.)$$

Hence this general theorem; "Let AB, AC, AD "*be three co-ordinates issuing at right angles from* "*any point* A, *and let* AM *be taken equal to* X, "AL *equal to* Y, *and* AN *to* Z. *Complete the* "*parallelopipedon, whose sides are* AM, AL, AN; "*then its diagonal* AG $(= \sqrt{X^2 + Y^2 + Z^2})$ [a] *will be* "*the distance of the common center of gravity of the particles* p, p', π, π', &c. *from the point* A;" the actual position of AG with respect to the planes passing through Bb, Cc, or Dd being determined by the signs of X, Y, and Z; for AM, AL, AN must be measured along the lines AB or Ab, AC or

(a) See Art. 6. page 82.

or Ac, AD or Ad, according as those signs are *positive* or *negative*.

5. If the particles lie in the plane passing through Bb, Cc, then $Z=0$, and the position of their common center of gravity will be determined by taking $AM=X$, $AL=Y$, and completing the parallelogram $ALKM$; for the *diagonal* (AK) of this parallelogram $(=\sqrt{X^2+Y^2})$ will then be the distance of that center from the point A. If the particles lie in *the same right line* (Bb), then Y also $=0$, and the distance of their common center of gravity from the point A is determined by taking $AM=(\sqrt{X^2}=)$

$$\frac{p\times pA+p'\times p'A+\&c.-\pi\times\pi A-\pi'\times\pi'A-\&c.}{p+p'+\&c.+\pi+\pi'+\&c.}$$ [a]

which coincides with the rule for finding the common center of gravity of any number of bodies placed in a right line, as exhibited in Exam. 5. page 107.

6. Suppose now the particles p, p', π, π', &c. to be all *equal to each other*, and *symmetrically* [b] disposed

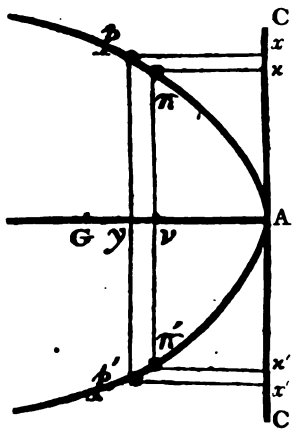
(a) For in this case px, $p'x'$, $\pi\kappa$, &c. becomes pA, $p'A$, πA, &c.

(b) By the *symmetrical* disposition of the particles with respect to the planes, is meant, that the *same number* of particles are placed at the *same distances* on each side the planes.

posed with respect to the planes passing through Cc, Dd; Bb, Dd; Bb, Cc, respectively; then $p \times px = \pi \times \pi x$, &c.; $p \times py = p' \times p'y'$, &c.; $p \times pz = p' \times p'z'$, &c; $\therefore$ (since the positive and negative factors in the numerators of the fractions expressing the values of X, Y, Z, page 133, *destroy each other*) $X = 0$, $Y = 0$, $Z = 0$; consequently $AG = (\sqrt{X^2 + Y^2 + Z^2})^2 = 0$. In regular solid bodies of uniform density, such as *cubes*, *spheres*, *cylinders*, &c. the particles of which they are composed are *symmetrically* placed with respect to three planes intersecting each other at right angles, and passing through their center of magnitude; in all these cases therefore (supposing the co-ordinates to issue from that center) $AG = 0$, or *the center of gravity coincides with the center of magnitude*, as we observed in the former part of this Lecture. In regular *plane* figures, such as *squares*, *circles*, *polygons inscribed in circles*, &c, &c, $Z = 0$, and their particles are *symmetrically* disposed with respect to two right lines intersecting each other at right angles in their centers of magnitude, $\therefore$ in this case also $X = 0$ and $Y = 0$, and their centers of gravity coincide with their center of magnitude. If the particles be *uniformly* arranged along a *right line*, and A be taken in the *middle of the line*, then will $X = 0$, $Y = 0$, $Z = 0$, and A will be the center of gravity.

7. If

7. If the equal particles p, π; p', π'; &c. be placed in corresponding parts of the curve pAp', whose *axis* coincides with the co-ordinate AB, and *bisects the parallel ordinates* $\pi\pi', pp'$, &c. then since they are in the *same plane*, $Z=0$; and because they are *symmetrically* placed with respect to the co-ordinate AB, $Y=0$; the center of gravity (G) therefore is determined by finding the value of X, which in this case is equal to $\frac{p\times px+\pi\times\pi x+p'\times p'x'+\&c.}{p+\pi+p'+\&c.}$, the signs of all the terms being *positive*, because the particles all lie on the *same side* of the plane passing through Cc. This leads to an easy investigation of the common *fluxional* expression for finding the center of gravity of any *curve line* or *curvilinear area*, and the *surface* or *solid* generated[a] by the revolution of such curve line or area about its axis.



8. For let the *abscissa* $Av=x$, *ordinate* $\pi v=y$, arc

(a) Or rather a *portion of matter* of an uniform density in the *shape* of such curve line, area, solid, &c. &c.

arc $A\pi = z$, then $\pi x = \pi' x' = A\pi = x$, $\therefore$ the particles π, π' are placed at the *same distance* from the co-ordinate AC. Hence,

For the Curve.

Since each particle π, π', p, p', &c. may be considered as a small part of the *curve*, it may be represented by the *fluxion of* z; the *numerator* of the fraction expressing the value of X will therefore be the *sum of the products arising from multiplying the fluxion of the curve into the abscissa* (x), which (adopting the fluxional *notation*) is the *fluent of* $x\dot{z}$; the *denominator* of the fraction is the *sum of the particles* π, π', p, &c. which may be expressed either *by* z, or by the *fluent of* $\dot{z}$. The general fluxional equation for determining the distance of the center of gravity of any curve line $\pi A \pi'$ from its vertex A, is therefore $X = \frac{f.x\dot{z}}{f.\dot{z}} = \frac{f.x\dot{z}}{z}$.

For the Area.

Suppose now a *line of particles* to extend from π to π' (which may be considered as representing the *fluxion of the area*), then the *common distance* of this line of particles from the co-ordinate AC will be (x), and the *numerator* of the fraction will be the *sum of the products arising from multiplying the fluxion of the area into the abscissa* (x), which is fluent of $y\dot{x} \times x$; the *denominator* is the *sum of*

all the lines of particles, or the *fluent of* $y\dot{x}$; hence $X = \frac{f.yx\dot{x}}{f.y\dot{x}} = \frac{f.yx\dot{x}}{\frac{1}{2p}\text{Area } pAp'}$.

For the Surface generated.

If the whole figure revolves round its axis AZ, then any particle (π) will generate a *ring of matter* whose distance from the plane passing through AC is (x), and which may be considered as the *fluxion of the curve surface* ($2py\dot{z}$)[a]; in this case, therefore, $X = \frac{\text{fluent of } x \times 2py\dot{z}}{\text{fluent of } 2py\dot{z}} = \frac{f.xy\dot{z}}{f.y\dot{z}}$.

For the Solid generated.

In the revolution of the figure, the *line of particles* $\pi\pi'$ will generate a *lamina* of matter of a circular form, whose distance from AC is (x) as before, and which may be considered as the *fluxion of the solid* ($py^2\dot{x}$)[b]; hence $X = \frac{\text{fluent of } x \times py^2\dot{x}}{\text{fluent of } py^2\dot{x}} = \frac{f.y^2x\dot{x}}{f.y^2\dot{x}}$. Let us now apply these expressions to a few actual examples.

EXAM.

(a) Here p = circumference of a circle whose diameter is unity = 3.1415, &c. ∴ the circumference of a circle whose diameter is $2y = 2py$, and the matter in the *ring* $= 2py\dot{z}$.

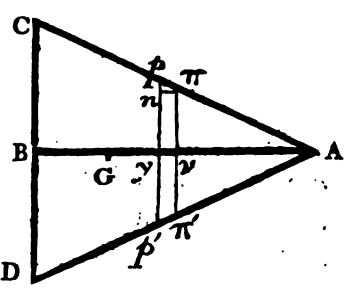
(b) The *area* of this lamina of matter is py^2, and its *thickness* ($\dot{x}$), the *content* of it therefore is $py^2\dot{x}$.

EXAM. 1.

To find the center of gravity in all these cases, when pAp' *is a* straight line.

When the curve pAp' becomes a *straight line*, the figure itself becomes an *isosceles triangle*, in which let $AB=a$, $AC=b$, $BC=c$, $A\nu=x$, $\pi\nu=y$, $A\pi=z$, and therefore $\nu y=\pi n=\dot{x}$, $pn=\dot{y}$, $p\pi=\dot{z}$. We shall consider each case separately.

1. For the *sides* AC, AB; since $AB\,(a) : AC\,(b) :: \pi n\,(\dot{x}) : \pi p\,(\dot{z})$, we have $\dot{z}=\frac{b\dot{x}}{a}$, $\therefore X=\frac{\int x\dot{z}}{\int \dot{z}}$ $=\frac{\frac{\int bx\dot{x}}{a}}{\int\frac{b\dot{x}}{a}}=\frac{bx^2}{2a}\times\frac{a}{bx}=\frac{x}{2}$; hence for the *whole triangle* $AG=\frac{1}{2}AB$.



2. For the *Area*. Since $AB\,(a) : BC\,(c) :: A\nu\,(x) : \pi\nu\,(y)$, we have $y=\frac{cx}{a}$; $\therefore X=\frac{\int yx\dot{x}}{\int y\dot{x}}=$ $\frac{\int\frac{cx^2\dot{x}}{a}}{\int\frac{cx\dot{x}}{a}}=\frac{cx^3}{3a}\times\frac{2a}{cx^2}=\frac{2x}{3}$; $\therefore AG=\frac{2}{3}AB$.

3. For

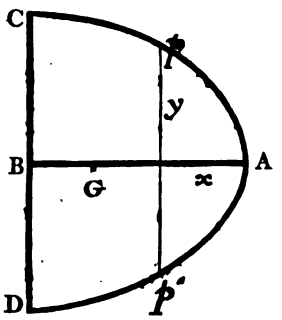
3. For the *Surface* generated. Here $X=\frac{f.xy\dot{z}}{f.y\dot{z}}$ $=\frac{f.\frac{bc}{a^2}\times x^2\dot{x}}{f.\frac{bc}{a^2}\times x\dot{x}}=\frac{bcx^3}{3a^2}\times\frac{2a^2}{bcx^2}=\frac{2x}{3}$; $\therefore$ for the whole *conical surface*, $AG=\frac{2}{3}AB$ the *same* as for the *area*.

4. For the *Solid*, or Cone. $X=\frac{f.y^2x\dot{x}}{f.y^2\dot{x}}=\frac{f.\frac{c^2}{a^2}x^3\dot{x}}{f.\frac{c^2}{a^2}x^2\dot{x}}$ $=\frac{c^2x^4}{4a^2}\times\frac{3a^2}{c^2x^3}=\frac{3x}{4}$; $\therefore AG=\frac{3}{4}AB$.

EXAM. 2.

To find the center of gravity of the area *and* solid *generated, when the curve pAp′ is a* Parabola.

As the fluent expressing the length of a parabolic *arc* in terms of its abscissa or ordinate contains *logarithmic* quantities, it is difficult to find the value of X, in the *first* and *third* cases, but it is very easily obtained for the *area* and *solid* generated. If $a=$ the *latus-rectum* of the Parabola, then $ax=y^2$;



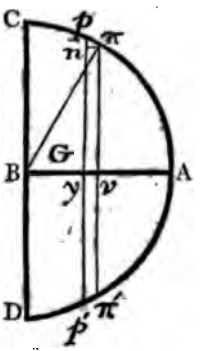
1. For the *Area*, $X=\frac{f.yx\dot{x}}{f.y\dot{x}}=\frac{f.a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}}{f.a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}}=\frac{2a^{\frac{1}{2}}x^{\frac{5}{2}}}{5}$ $\times\frac{3}{2a^{\frac{1}{2}}x^{\frac{3}{2}}}=\frac{3x}{5}$; $\therefore$ for the whole parabola $AG=\frac{3}{5}AB$.

2. For

2. For the *Solid.* $X=\frac{\int y^2 x\dot{x}}{\int y^2\dot{x}}=\frac{\int ax^2\dot{x}}{\int ax\dot{x}}=\frac{ax^3}{3}\times\frac{2}{ax^2}=\frac{2x}{3}$; ∴ for the whole paraboloid $AG=\frac{2}{3}AB$.

EXAM. 3.

To find the center of gravity of the arc, area, surface, *and* solid, *when the curve pAp' is a* circle.



Let B be the center of the circle, and let the radius $AB=B\pi=r$; and since $A\nu=x$, $B\nu\,(=AB-A\nu)=r-x$; now by similar triangles $p\pi n$, $B\pi\nu$, we have $p\pi\,(\dot{z}):pn\,(\dot{y})::B\pi\,(r):B\nu\,(r-x)$, ∴ $r\dot{y}=r\dot{z}-x\dot{z}$, and $x\dot{z}=r\dot{z}-r\dot{y}$.

1. For the *Arc.* $X=\frac{\int x\dot{z}}{\int\dot{z}}=\frac{\int r\dot{z}-\int r\dot{y}}{\int\dot{z}}=\frac{rz-ry}{z}=r-\frac{ry}{z}$; ∴ for the *semicircle*, $AG=r-\frac{r^2}{z}$ = (if $r=1$) $1-\frac{1}{\text{arc}\,AC}=1-\frac{1}{1.57\ \&c.}$ [a] $=\frac{.57\ \&c.}{1.57\ \&c.}=.363\ \&c.$

2. For

(a) The circumference of a circle whose *diameter* is 1= 3.1415 &c. ∴ circumference of a circle whose *radius* is 1=6.282, &c. and a *quadrant* of this latter circumference = 1.570, &c.

2. For the *Area*. Since $y^2=2rx-x^2$, $x^2=2rx-y^2$, $\therefore x\dot{x}=r\dot{x}-y\dot{y}$, and $yx\dot{x}=ry\dot{x}-y^2\dot{y}$; hence $X=\frac{f.yx\dot{x}}{f.y\dot{x}}=\frac{r\times f.y\dot{x}-f.y^2\dot{y}}{f.y\dot{x}}=\frac{r\times \text{area } A\pi\nu-\frac{1}{3}y^3}{\text{area } A\pi\nu}$ $=r-\frac{y3}{3 \text{ area } A\pi\nu}$; $\therefore$ for the *semicircle* $AG=$ $r-\frac{r3}{3 \text{ quad. } ABC}=$ (if $r=1$) $1-\frac{1}{3\times 1.57 \text{ &c.}}=$ $1-\frac{1}{4.71 \text{ &c.}}=\frac{3.71 \text{ &c.}}{4.71 \text{ &c.}}=.787$ &c.

3. For the *Surface*. By sim. $\Delta^s f\pi n$, $B\pi\nu$, $p\pi\,(\dot{z})$ $:\pi n\,(\dot{x})::B\pi\,(r):\pi\nu\,(y)$, $\therefore y\dot{z}=r\dot{x}$; hence $X=\frac{f.xy\dot{z}}{f.y\dot{z}}=\frac{f.rx\dot{x}}{f.r\dot{x}}=\frac{rx^2}{2rx}=\frac{x}{2}$; and $\therefore$ for the *hemisphere* $AG=\frac{1}{2}AB$.

4. For the *Solid*. $X=\frac{f.y^2x\dot{x}}{f.y^2\dot{x}}=\frac{f.2rx^2\dot{x}-f.x^3\dot{x}}{f.2rx\dot{x}-f.\dot{x}^2\dot{x}}$ $=\frac{\frac{2}{3}rx^3-\frac{1}{4}x^4}{rx^2-\frac{1}{3}x^3}=\frac{8rx-3x^2}{12r-4x}$; $\therefore$ for the *hemisphere* $AG=\frac{5}{8}AB$.

I might now proceed to apply these fluxional theorems to finding the centers of gravity of segments of the *ellipse*, *hyperbola*, and other Algebraic curves,[a] or of the surfaces and solids derived from them;

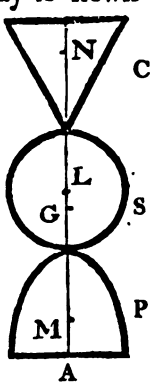
(a) By *Algebraic* curves are meant those in which the relation between the abscissa (x) and ordinate (y) is defined by an *algebraic equation*.

them; but questions of this kind abound in all the treatises upon the prineiples of fluxions; I shall therefore conclude this Lecture with a few examples, for the illustration of the method of finding the center of gravity of a body or system of bodies from the equation $AG = \sqrt{X^2 + Y^2 + Z^2}$.

EXAM. 4.

A body of the form exhibited in the annexed figure (in which P represents a paraboloid, *S a* sphere, *and C a* cone, *of equal heights) is hewn out of a cylinder of an uniform density; it is required to find its center of gravity.*

If the cylinder out of which this body is hewn be divided into three parts, each of which is represented by *unity*, then will $P = \frac{1}{2}$, $S = \frac{2}{3}$, and $C = \frac{1}{3}$,[b] and consequently this body : its *circumscribing cylinder* :: $\frac{1}{2} + \frac{2}{3} + \frac{1}{3}$: 3 :: 1 : 2, i. e. it is equal to *half* its circumscribing cylinder. Now this is evidently a case in which $Y = 0$, $Z = 0$, for the figure is *symmetrical* about its axis. Let M be the center of gravity of the *paraboloid*, L of the *sphere*, and N of the *cone*; then if the common height of these solids be represented by unity, AM will



(b) For the *paraboloid*, *sphere*, and *cone*, are respectively *one half*, *two thirds*, and *one third* their circumscribing cylinder.

will be $=\frac{1}{3}$, $AL=\frac{3}{2}$, and $AN=\frac{11}{4}$.[a] Conceive these three bodies to be collected in their respective centers of gravity, then, by Exam. 5. page 107, we shall have

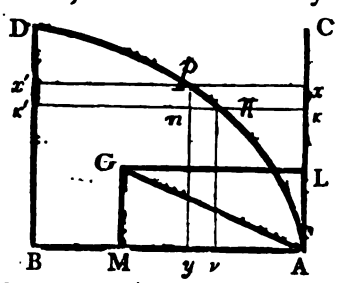
$$X \text{ or } AG=\left(\frac{P\times AM+S\times AL+C\times AN=}{P+S+C}\right)$$

$$\frac{\frac{1}{2}\times\frac{1}{3}+\frac{2}{3}\times\frac{3}{2}+\frac{1}{3}\times\frac{11}{4}}{\frac{1}{2}+\frac{2}{3}+\frac{1}{3}}=\frac{2+12+11}{6+8+4}=\frac{25}{18}=1\frac{7}{18},$$ *the height of the body itself being* 3.

EXAM. 5.

To find the center of gravity of the semi-Parabola *ABD.*

Since *ABD* is a *plane*, $Z=0$, and X has already been found (in Exam. 2.) $=\frac{3}{5}AB$; but since it is not *symmetrical* about the axis *AB*, we must find the value of Y. Let $AB=b$, then By or $px=b-x$; and reckoning the *fluxion of the figure* from *AB*, that fluxion will be expressed by (the parallelogram $px'\kappa' n$) $\overline{b-x}\times\dot{y}$; the *numerator* of the fraction therefore,



[a] For $AM=\frac{1}{3}$ of the axis by Case 2. Exam. 2. L is in the *center of the sphere*, $\therefore AL=1+\frac{1}{2}=\frac{3}{2}$; and M is $\frac{3}{4}$ from the vertex of the Cone by Exam. 1, $\therefore AN=2+\frac{3}{4}=\frac{11}{4}$.

therefore, which expresses the value of Y ($= f\dot{} y \times$ fluxion of the Figure) $= f\dot{} y \times \overline{b-x} \times \dot{y} = f\dot{} \overline{b-x} \times y\dot{y}$; hence $Y = \frac{f\dot{} \overline{b-x} \times y\dot{y}}{f\dot{} \overline{b-x} \times \dot{y}} = \frac{f\dot{} \overline{b-x} \times \frac{1}{2}a\dot{x}}{f\dot{} \overline{b-x} \times \frac{1}{2}a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}}$ [b]

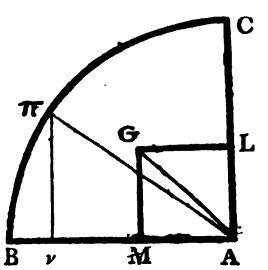
$= \frac{f\dot{} \frac{1}{2}ab\dot{x} - f\dot{} \frac{1}{2}ax\dot{x}}{f\dot{} \frac{1}{2}a^{\frac{1}{2}}bx^{-\frac{1}{2}}\dot{x} - f\dot{} \frac{1}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}} = \frac{\frac{1}{2}abx - \frac{1}{4}ax^2}{a^{\frac{1}{2}}bx^{\frac{1}{2}} - \frac{1}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}}$;

$\therefore$ for the whole parabola (when $x=b$) $Y = \frac{\frac{1}{2}ab^2 - \frac{1}{4}ab^2}{a^{\frac{1}{2}}b^{\frac{3}{2}} - \frac{1}{3}a^{\frac{1}{2}}b^{\frac{3}{2}}} = \frac{\frac{1}{4}ab^2}{\frac{2}{3}a^{\frac{1}{2}}b^{\frac{3}{2}}} = \frac{3}{8}a^{\frac{1}{2}}b^{\frac{1}{2}} =$ (for $BD^2 = a \times AB = ab$, and $\therefore BD = a^{\frac{1}{2}}b^{\frac{1}{2}}$) $\frac{3}{8}BD$. Take therefore $AM = X = \frac{3}{5}AB$, and $AL = Y = \frac{3}{8}BD$, and complete the parallelogram $AMGL$, then $AG = \sqrt{X^2 + Y^2}$ is known in terms of the given lines AB, BD.

EXAM. 6.

To find the center of gravity of the quadrant *of a circle ABC.*

Here it will be better to let A be in the *center of the circle;* then it is evident, since the figure is *symmetrically* placed with respect to the co-ordinates AB, AC, that X will be *equal* to Y. Let $A\nu = x$, $\pi\nu = y$, $A\pi = r$, then $x^2 + y^2 = r^2$, and $2x\dot{x} + 2y\dot{y} = 0$, or $x\dot{x} = -y\dot{y}$; $\therefore f\dot{} yx\dot{x} = f\dot{} -y^2\dot{y}$



$= -\frac{1}{3}y^3 +$ Cor.; when $f\dot{}=0$, $y=r$, $\therefore$ Cor. $= \frac{1}{3}r^3$, and $f\dot{} yx\dot{x}$

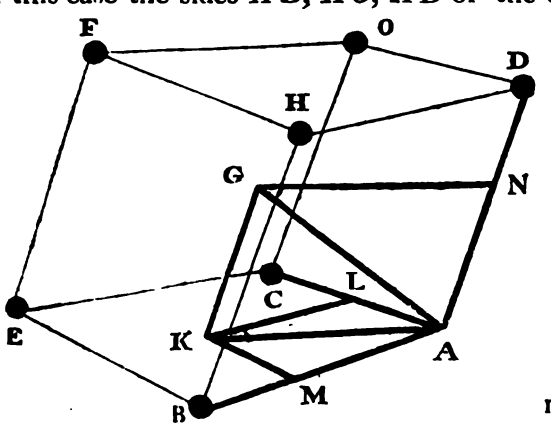
(b) The equation to the curve (see Exam. 2.) is $ax = y^2$, $\therefore a\dot{x} = 2y\dot{y}$, and $y\dot{y} = \frac{1}{2}a\dot{x}$; also $y = \sqrt{ax}$, $\therefore \dot{y} = \frac{1}{2}ax^{-\frac{1}{2}}\dot{x}$.

$\int y x \dot{x} = \frac{1}{3} r^3 - \frac{1}{3} y^3$; hence $X = \frac{\int y x \dot{x}}{\int y \dot{x}} = \frac{\frac{1}{3} r^3 - \frac{1}{3} y^3}{\text{area } AvC}$; and for the whole quadrant (when $y = 0$), $X = \frac{\frac{1}{3} r^3}{\text{quad. } ABC}$; let $r = 1$, then $X = \frac{1}{3 \text{ quad. } ABC}$ $= \frac{1}{3 \times .785}$ [a] $= \frac{1}{2.355} = .424$; but $X = Y$, $\therefore AG = \sqrt{X^2 + Y^2} = \sqrt{2X^2} = X\sqrt{2}$; take $\therefore AM$ and AL, each equal to .424, and complete the parallelogram $AMGN$, then $AG = .424\sqrt{2}$ $= .699 =$ distance of the center of gravity from the center A.

EXAM. 7.

Seven equal bodies are placed with their centers of gravity in the angles of the cubical space ABCDF; it is required to find the distance of their common center of gravity from the other angle A.

In this case the sides AB, AC, AD of the cube



may

[a] Area of a circle whose radius = 1 is 3.1415 &c. $\therefore$ a *quadrant* of this circle = .785 &c.

may represent the *co-ordinates* issuing from the point *A*. Let the *side of the cube* $= a$, and conceive the bodies to be collected in their respective centers of gravity; then, since the bodies *C, O, D* are placed in the plane *ACOD*, their distance from that plane is *equal to* 0, and consequently *three* of the factors in the *numerator* of the fraction expressing the value of X become equal to 0, $\therefore$ we shall have

$$AM = X = \frac{\overline{B+E+F+H} \times a}{\text{sum of the bodies}} = \text{(since the bodies are equal)} \ \frac{4a}{7}.$$

For the same reason, since the bodies *B, H, D* are placed in the plane *ABHD*, we have

$$AL = Y = \frac{\overline{E+F+O+C} \times a}{\text{sum of the bodies}} = \ldots\ldots \frac{4a}{7}.$$

And, since the bodies *B, E, C* are placed in the plane *ABEC*, we have

$$AN = Z = \frac{\overline{H+F+O+D} \times a}{\text{sum of the bodies}} = \ldots\ldots \frac{4a}{7}.$$

since $AG = \sqrt{X^2 + Y^2 + Z^2} = \sqrt{\frac{48a^2}{49}} = \frac{4\sqrt{3}}{7} \times a.$

In finding the general fluxional expression for the center of gravity of a *curve* and an *area* in page 137, we have calculated it only for its distance (X) from the axis *AC*, which answers the purpose for the *whole* curve or area, since we know from other circumstances that the center of gravity lies in the *axis AB*. If it be required to find the center

center of gravity of *half* the curve (viz Ap) or *half* the area (viz Apy), we must find the value of (Y), which in this case will be $\frac{f.y\dot{z}}{z}$ for the *curve*, and (supposing the *fluxional area* πvyp to be collected into its center of gravity) $\frac{f.y\dot{x}\times\frac{1}{2}y}{Apy}$ or $\frac{f.y^2\dot{x}}{2Apy}$ for the *area*[a]. (Fig. in page 136.)

Since $Y=\frac{f.y\dot{z}}{z}$ for the *curve*, and $Y=\frac{f.y^2\dot{x}}{2Apy}$ for the area, let $p=3.141$, &c.; then for the *curve*, $Y=\frac{f.2py\dot{z}}{2pz}$; and for the *area*, $Y=\frac{f.py^2\dot{x}}{2p\times Apy}$; hence $f.2py\dot{z}=2pY\times z=$ *the circumference of a circle whose radius is* $Y\times z$; and $f.py^2\dot{x}=2pY\times Apy=$ *the circumference of a circle whose radius is* $Y\times Apy$; but $f.2py\dot{z}=$ the *surface* generated by the

(a) From this mode of expressing the value of (Y), the two Examples in pages 144, 145, might have been solved in a more simple manner. For, in the *Parabola*, $y^2=ax$, $\therefore Y=\frac{f.y^2\dot{x}}{2Apy}=\frac{f.ax\dot{x}}{2Apy}=\frac{ax^2}{4Apy}=$(when $x=AB$, $y=BD$) $\frac{BD^2\times AB}{\frac{3}{8}BD\times AB}=\frac{3}{8}BD$. For the *quadrant of a circle*, if $Br=x$, $y^2=r^2-x^2$, $\therefore Y=\frac{f.y^2\dot{x}}{2BC\pi r}=\frac{f.r^2\dot{x}-x^2\dot{x}}{2BC\pi r}=\frac{r^2x-\frac{1}{3}x^3}{2BC\pi r}=$(when $x=r$) $\frac{\frac{2}{3}r^3}{2ABC}=\frac{r^3}{3ABC}$.

the revolution of the curve Ap round its axis AB, and $\int p y^2 \dot{x}$ = content of the *solid* generated by the revolution of the area Afg round the same axis. We are thus furnished with a method of ascertaining the value of the *surfaces* and *solids* generated by the revolution of Algebraic curves about their axis, when *we have previously found the distance of the center of gravity of the generating curve or area from that axis.* It is called the *centrobaryc* method; but in the present advanced state of analytical science, it is a method very little used; it will be quite sufficient therefore if we shew its application in one or two well-known cases.

1. Let h = the *length*, r = the *breadth* of a right-angled *parallelogram*, and let it revolve about (h); *the distance of its center of gravity from* $h = \frac{1}{2}r$, $\therefore$ the *circumference of the circle* described by it $= 2p \times \frac{1}{2}r = pr$; hence the *cylinder* generated = *area of parallelogram* $\times 2pr = hr \times pr = phr^2 = pr^2 \times h =$ *area of its base* $\times$ *its height.*

2. Let the *parabolic* area Apy (Fig. in page 136) revolve about its axis AB; then *the distance of its center of gravity from* $AB = \frac{3}{8}y$, $\therefore$ the *circumference of the circle* described by it $= 2p \times \frac{3}{8}y = \frac{3}{4}py$; hence the *solid generated* $= \frac{3}{4}py \times$ area $Apy = \frac{3}{4}py \times \frac{2}{3}xy = \frac{1}{2}pxy^2 =$ *half the circumscribing cylinder.*

Lecture V.

ON THE COLLISION OF BODIES.

PREVIOUS to the investigation of the effects produced by the *collision* of bodies, it will be necessary to define what is meant by *hardness*, *elasticity*, &c. and to explain the manner in which that collision takes place.

XVIII.

Definitions, and preliminary observations.

1. *Hardness* is that property of body by which it resists the impression of other bodies which impinge upon it; and the *degree* of hardness is measured by the quantity of this resistance. If the resistance be so complete as to render it totally incapable of any impression, then a body is said to be *perfectly* hard.

2. *Elasticity* is that property by which the parts of a body, after they have yielded to the impression of other bodies, restore themselves to their former position;

position; if the force of restitution be *equal*[a] to the force of compression, then a body is said to be *perfectly* elastic; but if the force of restitution be *less* than that of compression, then it is *imperfectly* elastic, and the *degree* of elasticity will be measured by the proportion which these two forces bear to each other.[b]

3. All such substances as come under the denomination of *solid* bodies are possessed of one or other of these qualities, but not in a state of *perfection*; they are *all* of them *imperfectly hard*, and *few* of them without some *small degree of elasticity*.[c] Glass, ivory, marble, steel, &c. are amongst

(a) i. e. if the parts of the body return to their former position precisely in the *same time* and in the *same manner* as that in which they were displaced.

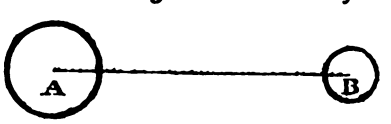
(b) It seems almost unnecessary to mention *softness* as a *distinct* property of matter, as it merely implies *imperfect hardness* in bodies void of elasticity.

(c) A common experiment for shewing that a substance is possessed of *elasticity*, is to suspend two spherical balls (of ivory for instance) from the same point by equal strings, and then, having stained one of them with ink, to bring them gently into contact with each other; a small black spot will thus be made upon the unstained ball just at the *point of contact*. If one of these balls be then made to *impinge* upon the other, that spot will be *enlarged*, whilst *the spherical figure of the ball remains the same*; from which it is inferred, that during the *act of collision* one or both the bodies must have been *compressed*, and that the parts so displaced must have afterwards *restored themselves to their former position*.

amongst the *most* elastic substances with which we are acquainted; and in some of these the proportion of their elasticity to *perfect* elasticity has been pretty nearly ascertained; thus in *glass*, the force of *restitution* has to the force of *compression* the ratio of about 15 : 16; in *steel* these forces are said to bear to each other the ratio of 5 : 9; but these ratios are subject to slight variations, according to the *shape or size* of the body.

4. In considering the changes which take place in the motion of bodies by the *act of collision*, it may be observed, that when a *perfectly hard* body impinges *directly*(a) upon another of the *same kind*, no other effect is produced than a communication of momentum from one to the other according to the principles of the *third law of motion*; for the two

(a) One body is said to impinge *directly* upon another, when the two bodies approach each other along the line which joins their centers of gravity; and if the bodies be of a *spherical shape* and of *uniform density*, then this line will pass through the point of contact. Thus suppose *A* and *B* to be two spherical bodies, whose centers of gravity are at the points *A*, *B*; join *AB*; then if the bodies approach each other in such a manner that their centers of gravity are always in the line *AB*, they are said to impinge upon each other by *direct* impact; and (by Euc. 3. 12.) the line *AB* (*A* and *B* being the centers of *magnitude* as well as *gravity*) will pass through their point of contact.



two bodies after impact will move on together as *one mass*, in the direction of the body which has the greater momentum.[b] But the contrary happens in the collision of *perfectly elastic* bodies; for although, during the act of *compression*, there is the same communication of momentum as in the former instance[c], yet during the act of *restitution* another collision (as it were) takes place, by which the bodies are protruded from each other, and after impact move on *separately*; each body having lost or gained *twice as much*

(b) For let the bodies before impact move in the *same* direction; then if the body which moves with the *greater velocity* has also the *greater momentum*, it will evidently drive the *slower body* before it; but if the *slower* body has the *greater momentum*, then the *swifter* body, having lost part of its velocity by the act of collision, will move on in *contact* with the slower body, as there is no cause operating to produce a *separation* of the bodies. If the bodies move in *opposite* directions, then, by the act of collision, the momentum of one of them will be *destroyed*, and it will be *driven back again* by that which has the *greater momentum*. If in this latter case the momenta be *equal*, it is evident that the two bodies would *rest* after impact.

(c) This communication of momentum from one body to the other will go on till the *velocities of the bodies are equal*; at which period the *act of compression ceases*, and the two bodies are under precisely the same circumstances as two *perfectly hard* bodies of the same dimensions; the momentum *lost* or *gained* will therefore be the *same*.

much momentum[a] as it would have done had they been *perfectly hard*. The effects produced by the collision of bodies which are *imperfectly hard* or *imperfectly elastic* will of course be of an *intermediate* nature; they will neither move on *together* after impact, as in the case of *perfectly hard* bodies, nor will they *recede from each other* with so great a velocity as if they had been *perfectly elastic*.

5. In the three following Sections we shall consider the circumstances which take place in the *direct* impact of bodies of this kind, beginning with the case of *perfectly hard* bodies, and then proceeding to such as are *perfectly elastic*; the theorems[b] which relate to the collision of bodies *imperfectly*

(a) Since the bodies are *perfectly* elastic, whatever force has been expended in the communication of momentum from one body to another during the act of *compression*, the *same* will be exerted (and for the *same purpose*) during the act of *restitution*; the *whole quantity* of momentum lost or gained will therefore be *double* of that which takes place in the collision of *perfectly hard* bodies. The same conclusion would follow, if one of the bodies was *perfectly hard* and the other *perfectly elastic*.

(b) To render the investigation of these theorems as simple as possible, we shall suppose the bodies to be *spheres* of an uniform density, moving *uniformly* along the line which joins their centers of gravity, and impelled by no other force than that which first puts them in motion along that right line; so that, in estimating their

imperfectly elastic will then be easily deduced, in all cases where the relation between perfect and imperfect elasticity can be expressed *arithmetically*.



XIX.

On the direct Impact of perfectly Hard Bodies.

It has already been observed, that when a *perfectly hard* body impinges upon another of the same kind, the two bodies will move on together after impact as *one mass*; in this case, therefore, we have only to find the *common velocity* after impact, and the *velocity lost or gained* by either of the bodies.

1. Let A and B be two perfectly hard bodies, a and b the respective *velocities* before impact; then will Aa represent the *momentum of* A before impact, and Bb the *momentum of* B.(c) If the bodies move

their momenta, we need only take into consideration the *quantity of matter* contained in them, and the *uniform velocity* with which they move.

(c) The *letters* A and B here represent the *quantities of matter* in the bodies A and B; since therefore their respective *velocities* are a and b, their *momenta* will be represented by Aa, Bb (Sect. 2. p. 6.)

move in the *same* direction, $Aa + Bb$ will represent the *sum* of their momenta before impact; but if the bodies move in *opposite* directions, then will the sum of their momenta be represented by $Aa - Bb$[a]. Now let v = their *common velocity* after impact, then $\overline{A+B} \times v$ will be the *momentum* of the *mass* after impact; but by the *third law of motion* no momentum is *lost* by the action (or collision) of the bodies upon each other[b]; hence $\overline{A+B} \times v = Aa \pm Bb$, $\therefore v = \frac{Aa \pm Bb}{A+B}$, where the *positive* or *negative* sign must be used according as the bodies move in the *same* or *opposite* directions before impact.

2. To find the velocity *lost by A* and *gained by B*, when the bodies move in the *same* direction, we have, *velocity lost by A* = velocity *before* impact $-$ velocity *after* impact $= a - \frac{Aa+Bb}{A+B} = \frac{B.\overline{a-b}}{A+B}$. The *velocity gained by B* = velocity *after* impact $-$ velocity

(a) Throughout this Lecture we shall always estimate the motion in the *direction of A*; whenever, therefore, the bodies move in *opposite* directions, if *A's* motion be reckoned *positive*, *B's* motion must be reckoned *negative*; hence, in the present instance, the *whole momentum* of the two bodies (or the *sum* of their momenta) is properly represented by $Aa - Bb$.

(b) See page 15.

velocity *before* impact $= \frac{Aa+Bb}{A+B} - b = \frac{A.\overline{a-b}}{A+B}$.

3. When the bodies move in *opposite* directions, the *velocity lost by* $A = a - \frac{Aa-Bb}{A+B} = \frac{B.a+b}{A+B}$; and the *velocity gained by* $B = \frac{Aa-Bb}{A+B} + b$[c] $= \frac{A.\overline{a+b}}{A+B}$.

4. If the bodies move in the *same* direction, then their *relative* velocity before impact (i. e. the velocity with which they *approach* each other) is $a-b$; and when they move in *opposite* directions, it is $a+b$; hence, if this relative velocity be expressed by (r), then, in *all cases*, the *velocity lost by* $A = \frac{Br}{A+B}$, and the *velocity gained by* $B = \frac{Ar}{A+B}$; whilst the *relative velocity*, therefore, continues the *same*, the velocity *lost by* A and *gained by* B will remain unaltered.

5. If

(c) If the bodies move in *opposite* directions *before* impact, and in *A's* direction *after* impact, then the *velocity gained by B* upon the whole (in *A's* direction) will be the *common velocity* + *its velocity before impact*.

5. If $B=A$, then we have,

$$\text{The common velocity} = \left(\frac{Aa \pm Bb}{A+B} =\right) \frac{A.a \pm b}{2A} = \frac{a \pm b}{2} = \frac{r}{2}.$$

$$\text{Velocity lost by } A = \left(\frac{Br}{A+B} =\right) \frac{Ar}{2A} = \frac{r}{2}.$$

$$\text{Velocity gained by } B = \left(\frac{Ar}{A+B} =\right) \frac{Ar}{2A} = \frac{r}{2}.$$

i. e. when the bodies are *equal*, the *common velocity*, and the velocities *lost by A* or *gained by B*, are all equal to *each other* and to *half the relative velocity*.

6. If B *rests* before impact, then $b=0$, and we have,

$$\text{Common velocity} = \frac{Aa}{A+B} \text{ (for } Bb=0) = (\text{if } A=B)\ \tfrac{1}{2}\, a.$$

$$(X)\ \text{Velocity lost by } A = \frac{Ba}{A+B} \text{ (for } Bb=0) = (\text{if } A=B)\ \tfrac{1}{2}\, a.$$

$$(Y)\ \text{Velocity gained by } B = \frac{Aa}{A+B} \text{ (for } Ab=0) = (\text{if } A=B)\ \tfrac{1}{2}\, a.$$

Now, *in general*, the velocity *lost by* $A=\frac{Br}{A+B}$, and the velocity *gained by* $B=\frac{Ar}{A+B}$; comparing, therefore, these expressions with the equations marked (X) and (Y), it is evident that the velocities lost by A, and gained by B, will be the same, whether *both bodies be in motion* before impact, or *A impinges upon B at rest* with a velocity equal to their *relative velocity* (r).[a]

(a) For from equation (X) it appears that if A impinges upon *B at rest* with velocity (r), then *velocity lost by* $A=\frac{Br}{A+B}$; but this is the velocity lost by A, when the bodies are *both moving* with a relative velocity $=r$; and the same reasoning applies to the *velocity gained by B*.

7. When the bodies move in *opposite* directions, $v=\frac{Aa-Bb}{A+B}$; if therefore $Aa=Bb$, $v=0$; i.e. if $A:B::b:a$ *(or the bodies move before impact with velocities inversely proportional to their quantities of matter)* then they will both *rest* after impact.

EXAM. 1.

A, weighing 10 *lbs. and moving with a velocity of* 3 *feet in a second, impinges upon B, which weighs* 7 *lbs. moving in the same direction, with a velocity of* 2 *feet in a second; to find the common velocity after impact, the velocity lost by A, and the velocity gained by B.*

Here $A=10$, $B=7$, $a=3$, $b=2$ $\Big\}$ $\therefore$ (by Art. 1.) *com.* $V'=\frac{Aa+Bb}{A+B}=\frac{30+14}{10+7}=\frac{44}{17}=2\frac{10}{17}$ feet in 1″.

V' *lost by* $A=\frac{B\times\overline{a-b}}{A+B}=\frac{7}{17}$ feet in 1″ (Art. 2.)

V' *gained by* $B=\frac{A\times\overline{a-b}}{A+B}=\frac{10}{17}$ feet in 1″.

EXAM. 2.

A, moving with a velocity (5), *impinges upon B moving in a contrary direction with a velocity* (3); *and, in the act of collision, A loses* $\frac{1}{3}$ *of its momentum; What are the relative magnitudes of A and B?*

Here

Here $a=5$, $b=3$ $\Big\}$ $\therefore$ *Velocity lost by* $A=\frac{B.\overline{a+b}}{A+B}$ (Art. 3.) $=\frac{8B}{A+B}$; hence the *momentum* lost by $A=\frac{8AB}{A+B}$. Now A's momentum before impact $=Aa=5A$; *one third* of which is $\frac{5A}{3}$; we have therefore $\frac{8AB}{A+B}=\frac{5A}{3}$, or $24B=5A+5B$; hence $19B=5A$, and $A:B::19:5$.

EXAM. 3.

A, weighing 7 lbs, impinges upon B weighing 4lbs, and moving, in A's direction, with a velocity of 5 feet in 1″; by the act of collision, B's velocity is trebled; What was A's velocity before impact?

Here $A=7$, $B=4$, $b=5$ $\Big\}$ $\therefore$ *Velocity gained by* $B=$ (by Art. 2.) $\frac{A\times\overline{a-b}}{A+B}=\frac{7\times\overline{a-5}}{11}$; but since, by the act of collision, B's velocity was *trebled*, its velocity *gained* $=2b=10$; hence $\frac{7\times\overline{a-5}}{11}=10$, or $7a-35=110$, $\therefore 7a=145$, and $a=20\frac{5}{7}$ feet in 1″.

And thus we might apply the principles laid down in this Section to the solution of a variety of examples of this kind; for let the *common velocity* after impact $=v$, *velocity lost by* $A=l$, and *velocity gained by* $B=g$; we have then,

$$v=\frac{Aa\pm Bb}{A+B};\quad l=\frac{B.\overline{a\pm b}}{A+B};\quad \text{and}\quad g=\frac{A.\overline{a\pm b}}{A+B};$$

if

if therefore, of the *five* quantities contained in each of these equations, any *three* be given, then the *proportion* of the other two may be found; or if any *four* be given, then the *actual value* of the fifth may be found.

XX.

On the direct Impact of perfectly Elastic Bodies.

Let the bodies A and B be *perfectly elastic*, then the velocity *lost by* A and *gained by* B will be *twice* that which it would have been on supposition that they were perfectly hard.[a] Moreover, as the bodies in this case will move on *separately* after impact, it will be necessary to find the velocities of A and B by *distinct operations*.

1. Let the bodies move before impact in the *same* direction;

$$\left.\begin{matrix}\text{then } V^y \text{ of } A \\ \text{after impact}\end{matrix}\right\} = \left.\begin{matrix}V^y \text{ before} \\ \text{impact}\end{matrix}\right\} - V^y \text{ lost}$$

$$= a - \frac{2B.\overline{a-b}}{A+B}^{(b)} = \frac{\overline{A-B}.a+2Bb}{A+B},$$

and

(a) See Note page 154, where it is shewn that in the impact of *perfectly elastic* bodies twice as much *momentum* is lost or gained as in the case of *perfectly hard* bodies; but since $M \propto Q \times V$, if Q be *given*, then $M \propto V$; *the velocity* lost therefore is twice as much as when the bodies are perfectly hard.

(b) For in *perfectly hard* bodies $l = \frac{B.\overline{a-b}}{A+B}$, and $g = \frac{A.\overline{a-b}}{A+B}$;

$\therefore$ in *perfectly elastic* bodies $\left\{\begin{matrix}l = \frac{2B.\overline{a-b}}{A+B}, \text{ and } g = \frac{2A.\overline{a-b}}{A+B}.\end{matrix}\right.$

$$\left.\begin{matrix}\text{and } V' \text{ of } B\\ \text{after impact}\end{matrix}\right\} = \left.\begin{matrix}\text{V}^{\text{y}} \text{ before}\\ \text{impact}\end{matrix}\right\} + \text{V}^{\text{y}} \text{ gained}$$

$$= b + \frac{2A.\overline{a-b}^{(a)}}{A+B} = \frac{\overline{B-A}.b+2Aa}{A+B}.$$

2. If the bodies move before impact in *opposite* directions, then b is *negative*, and

$$V' \text{ of } A \text{ after impact} = \frac{\overline{A-B}.a-2Bb}{A+B}.$$

$$V' \text{ of } B \text{ after impact} = \frac{\overline{A-B}.b+2Aa}{A+B}.$$

3. If $B=A$, then $A-B$ or $B-A$ are each equal to 0; $\therefore$ when the bodies move in the *same* direction before impact, the *velocity of A after impact* $= \left(\frac{2Bb}{A+B}=\frac{2Bb}{2B}=\right)b$; and *the velocity of B after impact* $= \left(\frac{2Aa}{A+B}=\frac{2Aa}{2A}=\right)a$. If they move before impact in *opposite* directions, then *velocity of A after impact* $= \left(\frac{-2Bb}{A+B}=\frac{-2Bb}{2B}=\right)-b$, and *the velocity of B after impact* $= \left(\frac{2Aa}{A+B}=\frac{2Aa}{2A}=\right)a$. Hence, in *all cases* when the bodies are *equal*, they move *after* impact with *interchanged velocities*.

4. If B *rests* before impact, then $b=0$; $\therefore$ the V' *of A after impact* $= \frac{\overline{A-B}.a}{A+B}$, and *velocity of B after*

(a) See Note (b) in the preceding page.

after impact $= \frac{2Aa}{A+B}$. If A be *greater* than B, then $\frac{\overline{A-B}.a}{A+B}$ is *positive*, $\therefore A$ moves *after* impact in the *same direction* as it did *before* with $V^y = \frac{\overline{A-B}.a}{A+B}$, and B *precedes* it with a velocity $= \frac{2Aa}{A+B}$ (which is *greater* than a). If A be *less* than B, then $\frac{\overline{A-B}.a}{A+B}$ is *negative*, $\therefore A$ is *reflected back* by its impact upon B with a velocity $= \frac{\overline{A-B}.a}{A+B}$; and B moves forward in *A's original direction* with a velocity $= \frac{2Aa}{A+B}$ (which is *less* than a). If A be *equal* to B, then $\frac{\overline{A-B}.a}{A+B} = 0$, & $\frac{2Aa}{A+B} = \frac{2Aa}{2A} = a$; i. e. if B rests *before* impact, then A will rest *after*; and B will move forwards in *A's direction* with *A's velocity before impact.*

5. If the bodies move with the *same* velocity in *opposite* directions, then $a = b$, and

A's velocity after impact $= \left(\frac{\overline{A-B}.a-2Bb}{A+B} = \right)\frac{\overline{A-3B}.a}{A+B}$.

B's velocity after impact $= \left(\frac{\overline{A-B}.b+2Aa}{A+B} = \right)\frac{\overline{3A-B}.a}{A+B}$.

Now

Now if A be *greater than* $3B$, then

$A-3B$ and $3A-B$ *will each be positive.*

If A be *equal to* $3B$, then

$$A-3B=0;\ \text{and}\ \frac{\overline{3A-B}.a}{A+B}=\frac{8Ba}{4B}=2a.$$

If A be *less than* $3B$, but *greater than* B, then

$A-3B$ is *negative*, and $3A-B$ is *positive*.

Hence this general theorem, "Let two perfectly "elastic bodies impinge upon each other with *equal* "*velocities* in *opposite directions*; then, if one body "be *more than treble* the other, *both bodies* will "move on after impact *in the direction of the greater* "*body*; if one body be *just treble* of the other, "then the *greater body* will remain *at rest* after "impact, and the *lesser body* will be *reflected back* "*again with twice the common velocity before impact*; "if one body be *greater than the other* but *less than* "*treble*, then *both bodies* will be *reflected back again* "*after impact*." If $A=B$, then $\frac{\overline{A-3B}.a}{A+B}=-a$, and $\frac{\overline{3A-B}.a}{A+B}=a$; i.e, the bodies after impact move with *interchanged* velocities according to Art. 3.

6. In Art. 7. of last Section, it was observed, that if $Aa=Bb$, then the two bodies *rest* after impact. In the case also of *perfectly elastic* bodies, approaching each other with *velocities inversely proportional to their quantities of matter*, the motion of both will be *destroyed* by the act of *compression*; but since, by

by the act of *restitution*, an equal quantity of motion is *generated*, the bodies after impact will move from each other in an *opposite direction*, with the same velocities with which they *approach* each other before impact.

7. Let r = the *relative* velocity of the bodies before impact, then, by Art. 4. of *perfectly hard* bodies, the *velocity lost by* $A = \frac{Br}{A+B}$, and *velocity gained by* $B = \frac{Ar}{A+B}$; let $\alpha = A$*'s velocity after impact*, and $\beta = B$*'s velocity after impact*, then the expressions in Art. 1, and 2, will stand thus

$$\alpha = a - \frac{2Br}{A+B}. \quad (X)$$

$$\beta = \pm b + \frac{2Ar}{A+B}. \quad (Y)$$

Subtract (Y) from (X), then

$$\alpha - \beta = a \mp b - \frac{2.\overline{A+B}.r}{A+B} = a \mp b - 2r,$$

where the *negative* or *positive* sign must be used according as the bodies move in the *same* or *opposite* directions.

If the bodies move in the *same* direction, then $r = a - b$, and

$$\alpha - \beta = a - b - 2a + 2b = b - a = -\overline{a-b}.$$

If the bodies move in opposite directions, then $r = a + b$, and

$$\alpha - \beta = a + b - 2a - 2b = -a - b = -\overline{a+b}.$$

From

From which we conclude, that when "two perfectly elastic bodies impinge upon each other, "their *relative velocity* will be the same both *before* "and *after* impact, but in an *opposite direction.*[a]"

8. By the third law of motion, the *whole quantity of momentum* is not altered by the impact of bodies upon each other, whether those bodies be *hard* or *elastic.* Now the sum of their momenta *before* impact $= Aa + Bb$; and (adopting the notation of the last article) the sum of their momenta *after* impact $= A\alpha + B\beta$; hence

$$Aa + Bb = A\alpha + B\beta^{[b]}$$

or $A.\overline{a-\alpha} = B.\overline{\beta-b}$; but by last Art. $\alpha - \beta = b - a$, $\therefore a + \alpha = \beta + b$,

Hence, by *multiplication*, $A.\overline{a^2-\alpha^2} = B.\overline{\beta^2-b^2}$,

or $Aa^2 + Bb^2 = A\alpha^2 + B\beta^2$; i. e.

"*the*

(a) Hence, when the bodies move before impact in the *same* direction, and A overtakes B; then, *after* impact, B will *recede* from A with the same velocity as that with which A *approached* B before impact, and consequently will move with a *greater* velocity than A. If the bodies move in *opposite* directions, then they will *recede from each other* after impact with the same velocity as they *approached each other* before impact.

(b) That the sum of their momenta is the same *before* and *after* impact, may also be thus proved;

By Art. 7. $\alpha = a - \frac{2Br}{A+B}$, $\therefore A\alpha = Aa - \frac{2ABr}{A+B}$.

$\beta = b + \frac{2Ar}{A+B}$, $\therefore B\beta = Bb + \frac{2ABr}{A+B}$.

Hence, by *addition*, $A\alpha + B\beta = Aa + Bb$; it is unnecessary in this article to take the case when b is *negative*, as our conclusion respects the *square* of B's velocity, which is the *same* whether b be $+$ or $-$.

" *the sum of the products arising from multiplying* " *each body into the square of its velocity* is the same " both *before* and *after* impact."

9. Let there be a row of *equal* perfectly elastic balls, *A, B, C,* &c. . . . *X* (Fig. 1.) placed contiguous to each other; then (by Art. 4.) if *A* be moved from its position and made to impinge upon *B*, it will *rest* after impact, and *B* will have a tendency to move on with *A*'s velocity; after the impact of *B* upon *C*, it will remain at *rest*, and *C* have a tendency to move



Fig. 1.

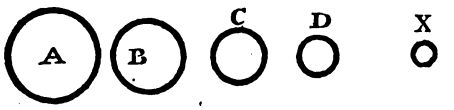


Fig. 2.

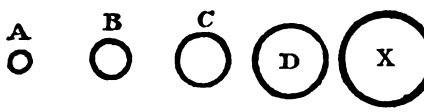


Fig. 3.

on with *A's* velocity; after the impact of *C* upon *D*, it will remain at *rest*, and *D* have a tendency to move on with the same velocity; and so the motion will be propagated through the whole row, and the *last* body *X* will move forward with the velocity of *A*, all the others remaining at rest. [c]. If the bodies

(c) If *two* bodies were moved from their position, and made to impinge upon the others, then the *two* last would move off; if *three* bodies were removed, the *three* last would move off; &c. &c.; for as the momentum is *doubled, trebled,* &c. towards *A*, it is evident that a *double, treble,* &c. mass will be put in motion towards *X*.

bodies *decrease* in magnitude (Fig. 2.) then, since A is *greater* than B, (by Art. 4.) the velocity communicated to B will be *greater* than that of A; and the velocity communicated from B to C greater than that of B; &c. &c. so that the *last* body will *move forwards* in the direction of A's motion with a velocity *much greater than that of* A [a], and the other bodies will follow it in such a manner that the velocity of each *succeeding* body shall be greater than that of the *preceding*. On the contrary, if the bodies *increase* in magnitude (Fig. 3.), since A is *less* than B, (by Art. 4.) the velocity communicated to B will be *less than that of* A, and A will be *reflected back* by B; for the same reason the velocity communicated from B to C will be *less than that of* B, and B will be *reflected back* by C; &c. &c.; so that in this case all the bodies will move *backwards* except the last, and that will move *forwards* in the direction of A's *original motion*, but with a velocity *much less* than that of A.

EXAM.

(a) The *greatest* velocity which can be generated by the propagation of motion through a row of perfectly elastic bodies, will be when those bodies are in *geometric* progression. For first take *three* bodies A, x, C, then (by Art. 4.) V_r communicated from A to $X = \frac{2Aa}{A+X}$; and when the body X impinges upon C at rest with $V_r = \frac{2Aa}{A+X}$, then the velocity communicated to C $= \left(\frac{2X}{X+C} \times \frac{2Aa}{A+X} =\right) \frac{4AaX}{A+X \times X+C}$, which latter quantity is to be a *maximum*; $\therefore \frac{A+X \times X+C}{4AaX}$ = a *minimum*, and

AX

EXAM. 1.

A and B are two perfectly elastic bodies; A weighing 6lbs, and moving with a velocity of 7 feet in a second, impinges upon B weighing 5lbs, and moving (in the same direction) with a velocity of 4 feet in a second; What are the velocities of A and B after impact?

Here $A=6$, $B=5$, $a=7$, $b=4$ $\Big\}$ $\therefore V^y$ *of A after impact* = (by Art. 1.) $\frac{\overline{A-B}.a+2Bb}{A+B}=\frac{7+40}{11}=4\frac{3}{11}$ feet per second. V^y *of B after impact* $=\frac{\overline{B-A}.b+2Aa}{A+B}=\frac{-4+84}{11}=7\frac{3}{11}$ feet per second.

Hence A and B move *after* impact in the *same direction* as they did *before* impact, A having *lost* a velocity of $2\frac{8}{11}$ feet per second, and B having *gained* a velocity of $3\frac{3}{11}$ feet per second.

EXAM.

$\frac{AX+AC+X^2+CX}{X}$, or $A+\frac{AC}{X}+X+C=min.$; hence $\frac{-AC\times\dot{X}}{X^2}+\dot{X}=0$, or $X^2=AC$, $\therefore X=\sqrt{AC}$. The velocity communicated from A, through B to C, will therefore be *greatest* when B is a *mean proportional between A and C*; for the same reason the velocity communicated from B through C to D will be *greatest* when C is a *mean proportional between B and D*; &c. &c. The *whole velocity* therefore communicated from A, through B, C, D, &c. to X, will be *greatest* when all the bodies A, B, C, D, &c. are in *geometric progression*.

EXAM. 2.

$A : B :: 3 : 5$; A and B move in opposite directions with velocities as $4 : 7$; What are the directions and velocities of A and B after impact?

Here $A=3$, $B=5$, $a=4$, $b=7$ } $\therefore V^y$ *of* A *after impact* = (by Art. 2) $\frac{\overline{A-B}.a-2Bb}{A+B} = \frac{-8-70}{8} = -9\frac{3}{4}$.

V^y *of* B *after impact* $= \frac{\overline{A-B}.b+2Aa}{A+B}$

$= \frac{-14+24}{8} = 1\frac{1}{4}$. Hence A and B are *reflected back* by impact (for $1\frac{1}{4}$ being *positive*, B moves after impact in *A's original direction*) with velocities which are to each other as $9\frac{3}{4} : 1\frac{1}{4}$, or as $39 : 5$.

EXAM. 3.

To determine the ratio of two perfectly elastic balls A and B, so that A, moving with a velocity of 5 feet in $1''$, and striking upon B at rest, shall, after impact, have a velocity which is to the velocity of $B :: 2 : 5$.

Here $a=5$; and (by Art. 4.) when A impinges upon B *at rest*, *A's velocity after impact* $= \frac{\overline{A-B}.a}{A+B}$, and *B's velocity after impact* $= \frac{2Aa}{A+B}$; hence $\frac{\overline{A-B}.5a}{A+B} : \frac{2Aa\times 5}{A+B} :: 2 : 5$, or $A-B : 2A :: 2 : 5$, $\therefore 5A-5B=4A$, or $A=5B$, $\therefore A : B :: 5 : 1$.

EXAM.

EXAM. 4.

A and B move in opposite directions; $A = 5B$, and $b = 3a$; How do the bodies A and B move after impact?

By Art. 2. *A's velocity after impact* $\left(= \frac{\overline{A-B}.a-2bB}{A+B}\right)$ $= \frac{4Ba-6Ba}{6B} = -\frac{1}{3}a$. *B's velocity after impact* $\left(= \frac{A-B.b+2Aa}{A+B}\right) = \frac{4Bb+\frac{10}{3}Bb}{6B} = \frac{11}{9}b$. Hence *A* and *B* are *reflected* back again with $\frac{1}{3}$d and $\frac{11}{9}$ths of their respective velocities before impact.

EXAM. 5.

Let there be a row *of bodies in* geometric progression *placed contiguous to each other, whose respective magnitudes are expressed by A, rA, r^2A, r^3A,* &c. . . $r^{n-1}A$ *(where n = the number of bodies); and let the* first *body impinge upon the* second *with the velocity (a); It is required to find the velocity of the* last *body, when the motion, thus communicated, has been propagated through all the intermediate bodies.*

By Art. 4, when *A* impinges upon *B at rest*, the velocity communicated to *B* is $\frac{2Aa}{A+B}$; in the present instance, therefore, the velocity communicated to the *second ball* (rA) will be $\frac{2Aa}{A+rA} = \frac{2a}{1+r}$;

$\frac{2a}{1+r}$; for the same *reason* the velocity communicated to the *third ball* will be $\frac{2rA}{rA+r^2A} \times \frac{2a}{1+r}$ [a] $= \frac{2^2a}{\overline{1+r}^2}$; and proceeding in this manner, it appears that the velocity of the *last* ball will be $\frac{2^{n-1}a}{\overline{1+r}^{n-1}}$, or

V^y of *first* ball : V^y of *last* $\left(:: a : \frac{2^{n-1}a}{\overline{1+r}^{n-1}}\right)$ $:: \overline{1+r}^{n-1} : 2^{n-1}$.

Cor. 1. Hence, if the *bodies* be in a geometric progression whose *common ratio is* r, the corresponding *velocities* will be in a geometric progression whose *common ratio is* $\frac{2}{1+r}$. The *former* of these progressions will moreover be an *increasing* or a *decreasing* one, and the *latter* a *decreasing* or an *increasing* one, according as r is *greater* or *less* than *unity*; which accords with what was shewn in Art. 9. of this Section.

Cor. 2. Let $\frac{2}{1+r} = \rho$; then if the series of *bodies* be represented by A, rA, r^2A, &c. . . . $r^{n-1}A$, the *corresponding velocities* will be represented by $a, \rho a, \rho^2 a,$

(a) The velocity of B after impact $= \frac{2Aa}{A+B}$ $= \frac{\textit{twice the striking body} \times \textit{its velocity}}{\textit{sum of the bodies}}$, which is a general expression for finding the velocity of the bodies in succession.

$\rho^2 a$, &c. . . $\rho^{n-1} a$. Let $\rho = \frac{1}{r^m}$, then $\rho^{n-1} = \frac{1}{r^{m.\overline{n-1}}}$; $\therefore$ V^y of *first* body : V^y of *last* body $\left(:: a : \frac{a}{r^{m.\overline{n-1}}}\right)$ $:: r^{m.\overline{n-1}} : 1$. Call the *last* body T, then (since $T = r^{n-1} A$) $A : T :: 1 : r^{n-1}$, $\therefore$ $r^{m.\overline{n-1}} : 1 :: T^m : A^m$; hence V^y of *first* body : V^y of *last* body $:: T^m : A^m$.

Cor. 3. Let the difference between any two successive bodies be very *small*, i. e. let $r = 1 \pm x$, where x is an *indefinitely small quantity*; then ρ $\left(= \frac{2}{1+r}\right) = \frac{2}{2 \pm x} = \frac{1}{1 \pm \frac{1}{2}x}$; but ρ is *also* equal to $\frac{1}{r^m}$; $\therefore \frac{1}{1 \pm \frac{1}{2}x} = \frac{1}{\overline{1 \pm x}^m} = \frac{1}{1 \pm mx + m.\frac{m-1}{2}x^2 \pm \&c.}$ $=$(rejecting x^2, x^3, &c.[a]) $\frac{1}{1 \mp mx}$; hence $m = \frac{1}{2}$; in this case, therefore, the velocity of the *first* body : V^y of the *last* :: $(T^m : A^m ::) \sqrt{T} : \sqrt{A}$.

EXAM. 6.

Let the bodies bear the same relation to each other as in the last example; It is required to find the number of intermediate *balls, that the velocity of the* last *ball may be increased to* m times *that of the* first *ball.*

Let

(a) See Alg. Sect. LV. on the *approximation of ratios.*

Let p = the number of *intermediate* balls, then $n = p + 2$, $\therefore n - 1 = p + 1$; now velocity of *last ball* $= \frac{2^{n-1}a}{\overline{1+r}^{n-1}} = \frac{2^{p+1}a}{\overline{1+r}^{p+1}}$: hence $\frac{2^{p+1}a}{\overline{1+r}^{p+1}} = ma$, or $\frac{2^{p+1}}{\overline{1+r}^{p+1}} = m$, $\therefore \overline{p+1} \cdot \log.\frac{2}{1+r} = \log. m$; from which we have $p + 1 = \frac{\log. m}{\log.\frac{2}{1+r}}$, and $p = \frac{\log. m}{\log. 2 - \log. \overline{1+r}} - 1$.

XXI.

On the direct Impact of imperfectly Elastic Bodies.

1. When *imperfectly* elastic bodies impinge upon each other, the same effect is produced during the act of *compression* as if the bodies were *perfectly* elastic; but in the act of *restitution* the velocity *lost by A* or *gained* by B is not *equal* to the velocity lost or gained during the act of *compression*, but is diminished in the ratio of $m : 1$ (if $m : 1$ represents the ratio of *imperfect* : *perfect* elasticity). The velocity lost by A therefore, during the act of *compression*, is $\frac{B.\overline{a-b}}{A+B}$; during the act of *restitution* it is $\frac{mB.\overline{a-b}}{A+B}$; hence, upon the whole, the *velocity lost*

lost by $A = \frac{\overline{1+m}.B.\overline{a-b}}{A+B.}$, and, for the same reason, the V^y *gained by* $B = \frac{\overline{1+m}.A.\overline{a-b}}{A+B.}$ (a).

2. The *velocity of A after impact* $= a - \frac{\overline{1+m}.B.\overline{a-b}}{A+B} = \frac{\overline{A-mB}.a+\overline{1+m}.Bb}{A+B}$. The *velocity of B after impact* $= b + \frac{\overline{1+m}.A\ \overline{a-b}}{A+B} = \frac{\overline{B-mA}.b+\overline{1+m}.Aa}{A+B}$. If $b=0$, or B *rests* before impact, then *A's* V^y *after impact* $= \frac{\overline{A-mB}.a}{A+B}$, and *B's V after impact* $= \frac{\overline{1+m}.Aa}{A+B}$. If $A-mB=0$ (or $A=mB$) then $\frac{\overline{A-mB}.a}{A+B}=0$; i.e. if A be *equal to* mB, and impinges upon B *at rest*, then A *rests* after impact, and B moves forward with a velocity $= \frac{\overline{1+m}.mBa}{mB+B} = ma$.

3. Let α, β represent *A's* and *B's* velocity after impact,

(a) To prevent a confusion of *signs*, we shall, through all the Articles of this Section, suppose the bodies to be moving in the *same* direction; as the expressions thus deduced are always convertible into such as apply to them when moving in *opposite* directions, merely by making the sign of (b) *negative*.

impact, then $\alpha = a - \frac{\overline{1+m}.B.\overline{a-b}}{A+B}$ (X)

$$\beta = b + \frac{\overline{1+m}.A.\overline{a-b}}{A+B} \quad (Y).$$

Subtract (X) from (Y), then

$$\beta - \alpha = b - a + \frac{\overline{1+m}.\overline{A+B}.\overline{a-b}}{A+B}$$

$$= b - a + \overline{1+m}.a - b \ = m.\overline{a-b}.$$

Hence the relative velocity *before* impact $(a-b)$: relative velocity *after* impact in an *opposite* direction $(\beta-\alpha)$:: $a-b$: $m.\overline{a-b}$:: 1 : m :: *perfect* : *imperfect* elasticity.

4. The sum of the products arising from multiplying each body into the square of its velocity *before* impact, is *greater* than the sum of the products arising from multiplying each body into the square of its velocity *after* impact.

For $Aa + Bb = A\alpha + B\beta$ (X) by Art. 8, p. 166; let $m = 1 - n$, then, by last Art.

$$\beta - \alpha = \overline{1-n}.\overline{a \quad b} = a - b - n.\overline{a-b} \quad (Y);$$

$\therefore$ from equation

(X) $A.\overline{a-\alpha} = B.\overline{\beta-b}$

(Y) $a + \alpha = \beta + b + n.\overline{a-b}$,

and by *multiplication*,

$$Aa^2 - A\alpha^2 = B\beta^2 - Bb^2 + nB.\overline{a-b}.\overline{\beta-b},$$

or $Aa^2 + Bb^2 = A\alpha^2 + B\beta^2 + nB.\overline{a-b}.\overline{\beta-b}.$

If the bodies move in the *same* direction, a and β are *each* of them greater than b; and if they move in *opposite*

opposite directions, b *itself is negative*; the quantity $nB.\overline{a-b}.\overline{\beta-b}$ is therefore *always positive*, consequently Aa^2+Bb^2 is *greater* than $A\alpha^2+B\beta^2$.

5. Let there be a *row* of imperfectly elastic bodies, $A, \frac{A}{m}, \frac{A}{m^2}$, &c. . . . $\frac{A}{m^{n-1}}$, in a geometrical progression, whose *common ratio* is $\frac{1}{m}$ (where n = the *number* of bodies, and $1 : m$ expresses the ratio of *perfect* to *imperfect* elasticity), placed *contiguous* to each other; and let the *first* body A impinge upon the second with the velocity $= a$; then all the *intermediate* bodies will remain at *rest* after impact, and the last body will move off with a velocity $= m^{n-1}a$. For let the intermediate bodies be B, C, D, &c. then $\frac{A}{m}=B$, $\therefore A=mB$; $\frac{A}{m^2}=C$, $\therefore \frac{A}{m}=mC$, i. e. $B=mC$; &c, &c.; but by Art. 2. when $A=mB$, and A impinges upon B at rest with velocity (a), A *rests* after impact, and B moves forward with a velocity (ma), which call (b); for the same reason, since $B=mC$, and B impinges upon C at rest with velocity $(b=ma)$, B will *rest* after impact, and C move forward with a velocity $(=mb=m^2a)$; and so on. When the motion therefore has been thus propagated through the whole row of bodies, all the *intermediate* bodies will remain at *rest*, and the *last* body will move forward with a velocity $= m^{n-1}a$.

6. If there be a row of elastic bodies, A, rA, r^2A, &c. . . $r^{n-1}A$, whose *degree* of elasticity and mode of *acting upon each other* is the same as before, then the bodies will all *move* after impact, (unless $r=\frac{1}{m}$) and the velocities with which they impinge upon each other will be represented by a, ρa, $\rho^2 a$, &c. . . $\rho^{n-1}a$, where $\rho=\frac{1+m}{1+r}$. For when A impinges upon rA at *rest*, the velocity communicated = (by Art. 2.) $\frac{\overline{1+m}.Aa}{A+rA}$ $=\frac{\overline{1.+m}.a}{1+r}$ (b); for the same reason, velocity communicated to *third* body $=\frac{1+m}{1+r}\times b=\frac{\overline{1+m}^2}{\overline{1+r}^2}\times a$, &c. &c. so that the velocity communicated to the *last* body $=\frac{\overline{1+m}^{n-1}}{\overline{1+r}}\times a$.

7. Let $r=m$, then $\rho=\frac{1+m}{1+r}=1$; $\therefore$ ρ and all *its powers* $=1$; i.e. when the *common ratio* by which the bodies *decrease* is the same *fraction* as that which expresses the *degree* of elasticity, the velocity communicated in each case will be that with which the *first* body struck the *second*, and with this velocity will the *last* body move off. So that in this case the same effect is produced upon the *last* body as when a row of *equal perfectly elastic* bodies are placed contiguous to each other; but the other bodies do not remain at rest after impact.

8. Since

8. Since, by Art. 3, $a = a - \frac{\overline{1+m}.B.\overline{a-b}}{A+B}$, we have $\frac{\overline{1+m}.B.\overline{a-b}}{A+B} = a - \alpha$, $\therefore 1+m = \frac{\overline{A+B}.\overline{a-\alpha}}{B.\overline{a-b}}$, and $m = \frac{\overline{A+B}.\overline{a-\alpha}}{B.\overline{a-b}} - 1$; hence, if the magnitudes and velocities of the bodies *before* impact be given, as also the velocity of A *after* impact, the ratio of *perfect* to *imperfect* elasticity may be determined. It is evident, that it might also be determined from *the equation expressing the velocity of B after impact.*

EXAM. 1.

Let $A = 4$, $B = 2$, $a = 3$, $b = 1$, *and perfect : imperfect elasticity* :: $1 : \frac{1}{2}$; *It is required to find the velocities of A and B after impact, when the bodies move before impact in the same direction.*

The *velocity of A after impact* $\left(= a - \frac{\overline{1+m}.B.\overline{a-b}}{A+B}\right)$ $= 3 - \frac{\frac{3}{2} \times 2 \times 2}{6} = 3 - 1 = 2$. The *velocity of B after impact* $\left(= b + \frac{\overline{1+m}.A.\overline{a-b}}{A+B}\right) = 1 + \frac{\frac{3}{2} \times 4 \times 2}{6} =$ $1 + 2 = 3$.

EXAM. 2.

A (= 5 *lbs.*) *impinges upon B* (= 3 *lbs.*) *at rest; B moves forward after impact with* $\frac{7}{8}$*ths the velocity of A before impact; What is the ratio of perfect to imperfect elasticity in these bodies?*

By

By Art. 2, *B's velocity after impact* $= \frac{\overline{1+m}.Aa}{A+B}$;

$\therefore$ in the present instance $\frac{\overline{1+m} \times 5a}{8} = \frac{7a}{8}$, or $5m+5=7$, and $5m=2$; hence $m=\frac{2}{5}$, and *perfect* : *imperfect* elasticity :: $1 : \frac{2}{5}$:: 5 : 2.

EXAM. 3.

A set of 5 *balls, imperfectly elastic, and in a geometrical progression whose common ratio is* 2, *are placed contiguous to each other; the force of elasticity is to the force of compression* :: 3 : 2. *Compare the velocity with which the first ball impinges upon the second, with the velocity communicated to the last.*

Since *perfect* : *imperfect* elasticity :: 3 : 2 :: 1 : $\frac{2}{3}$; by referring to Art. 6, we have $m=\frac{2}{3}$, $r=2$, $n=5$; $\therefore$ V^y of *first* ball : velocity of *last* ball (:: a : $\left|\frac{1+m}{1+r}\right|^{n-1} \times a$) :: $1 : \left|\frac{5}{9}\right|^4$:: $1 : \frac{625}{6561}$:: 6561 : 625.

EXAM. 4.

B is double of A; A's velocity before impact : *B's* V^y :: 3 : 2; *they move in the same direction, and by the act of collision A loses* $\frac{2}{5}$*ths of its velocity; What is the ratio of perfect to imperfect elasticity?*

Here

Here $B=2A$, $b=\frac{2}{3}a$, $a=\frac{3}{5}a$ $\Big\}$ $\therefore$ by Art. 8, $m\left(=\frac{\overline{A+B}.\overline{a-a}}{B.\overline{a-b}}-1\right)$

$=\frac{3A\times\frac{2}{5}a}{2A\times\frac{1}{3}a}-1=\frac{9}{5}-1=\frac{4}{5}$; hence

perfect : imperfect elasticity :: 1 : $\frac{4}{5}$:: 5 : 4.

XXII.

QUESTIONS FOR PRACTICE.

1. A and B are two *perfectly hard* bodies meeting in *opposite* directions; A is double of B, but moves with a velocity (a), which is only $\frac{2}{3}$ds the velocity of B; What is the *common velocity* after impact?

Answer . . . Com. V$^y=\frac{1}{6}$th a.

2. A : B :: 3 : 2, and the *velocity* of A (a) : velocity of B :: 5 : 4; they are *perfectly hard* bodies, and move before impact in the *same* direction; What is the velocity *lost by* A, and *gained by* B?

Ans. V^y lost by $A=\frac{2a}{25}$,

and V^y gained by $B=\frac{3a}{25}$.

3. A and B are two *perfectly hard* bodies moving in *opposite* directions, with velocities as 7 : 5; by the act of collision A loses $\frac{1}{4}$th of its momentum; What is the ratio of A : B?

Ans. A : B :: 41 : 7.

4. A

4. *A* and *B* are *perfectly hard*, and in the ratio of 6 : 5; *B* is moving in the *same* direction as *A*, with a velocity of 7 feet in a second; by the act of collision, *B*'s velocity is doubled; What was *A*'s velocity before impact?

Ans. $19\frac{5}{6}$ feet in 1″.

5. *A* and *B* are *perfectly elastic*, and in the ratio of 4 : 3; they are moving in the *same* direction with velocities as 5 : 4; What is the ratio of the velocities of *A* and *B* after impact?

Ans. V^y of *A* : V^y of *B* :: 29 : 36.

6. *A* and *B* are *perfectly elastic;* they are moving in *opposite* directions; *A* is treble of *B*, but *B*'s velocity is double that of *A;* How do these bodies move after impact?

Ans. *A* and *B* are *reflected* back again with $\frac{1}{2}$ and $\frac{5}{4}$ths of their respective velocities before impact.

7. There is a row of 6 *perfectly elastic* bodies in a *decreasing* geometric progression (whose common ratio is $\frac{1}{3}$) placed contiguous to each other; the first body impinges upon the second with a velocity of one foot per second; *With what velocity will the last body move off?*

Ans. With a velocity of $7\frac{19}{32}$ feet in 1″.

8. There is a row of perfectly elastic bodies in an increasing

increasing geometric progression whose common ratio is 3; a motion is propagated through them in the same manner as in last Example, and the last body moves off with $\frac{1}{64}$th the velocity of the first body; *What was the number of bodies?*

Ans. 7.

9. $A=2B$; *A's* velocity (a) : *B's* velocity (b) :: 3 : 2; and *perfect* : *imperfect* elasticity :: 4 : 3; they impinge upon each other, moving in the *same* direction; What are their respective velocities after impact?

Ans. *A's* velocity after impact $=\frac{29a}{36}$; *B's* $=\frac{19b}{12}$.

10. $A=3B$, and impinges upon B *at rest* with a velocity ($=a$); *A's* velocity *after* impact is $\frac{3}{5}$ths of its velocity *before* impact; What is the ratio of *perfect* : *imperfect* elasticity?

Ans. Perfect : imperfect elasticity :: 5 : 3.

11. Five *imperfectly elastic* balls (in which the force of compression : force of restitution :: 6 : 5) are placed contiguous to each other; these balls are in a geometrical progression, whose common ratio is $\frac{1}{2}$; the *first* ball impinges upon the *second* with a velocity of 3 feet in 1″; It is required to shew with what velocity the *last* body moves off.

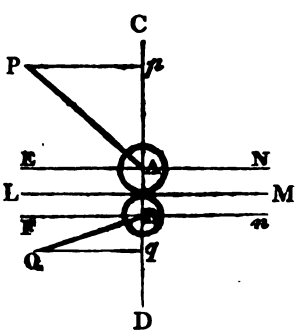
Ans. With a velocity of 6.69 feet in 1″.

XXIII.

On the oblique Impact of Hard and Elastic Bodies.

Hitherto we have considered the bodies as impinging upon each other *in the straight line which joins their respective centers of gravity*; we are now to estimate the effects produced by impact when the directions in which the bodies move *are inclined in some given angle to each other.*

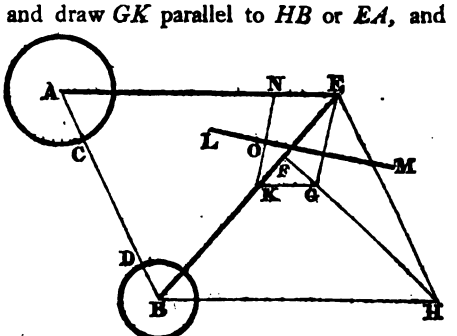
1. Let A and B be two bodies moving uniformly in the directions PA, QB, and let their motions be so adjusted, that PA, QB shall be described in the *same time*; then (since $V \propto S$, when T is given) PA, QB will represent their *velocities*. Let LM be the plane which touches both the bodies at their point of contact[a], and join their centers, A, B; then AB will be at right angles



(a) If the radii of these spherical bodies, together with their velocities and directions, be given, the position of this plane may be thus determined geometrically. Let the directions of the bodies meet in E; and let their velocities be such, that whilst A would describe AE, B would describe BF; and $\therefore$ V^y of A : V^y of B :: AE : BF. Complete the parallelogram $ABHE$, and join HF. With

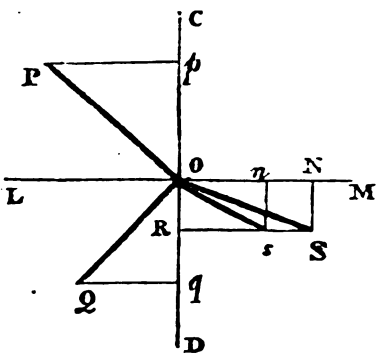
angles to LM. Produce AB both ways to C and D, and from P, Q let fall Pp, Qq perpendicular to CD; through A, B draw EN, Fn parallel to them, and consequently parallel to LM. PA, QB, which present the respective velocities of A and B, may be considered as resolved into two others, Pp, pA, and Qq, qB (Art. 7. page 51); of which Pp, Qq, being *parallel to* LM, are not affected by the collision of the bodies; if therefore AN, Bn be taken respectively equal to Pp, Qq, they will represent the velocities of the bodies in those directions *after impact*. The velocities of the bodies after impact, *in the direction* CD must be determined (according to

With center E and radius $=AC+BD$ describe a circular arc cutting HF in G; join EG, and draw GK parallel to HB or EA, and complete the parallelogram $EGKN$. By sim. Δ^s BH (or AE) : BF :: KG (or NE) : KF; since therefore NE : KF :: AE : BF :: V^y of A : V^y of B, NE and KF are spaces described in the *same time*, and, consequently, when A arrives at N, B will arrive at K; but $NK=EG=AC+BD$, $\therefore$ NK is the line which joins their centers at that time. Hence NK passes through the point of contact of the two bodies; and if NO be taken equal to AC, and, through O, LM be drawn at right angles to NK, it will give the position of a plane touching both the bodies at that point.



to the circumstances under which the collision takes place) by the rules laid down in the preceding sections.

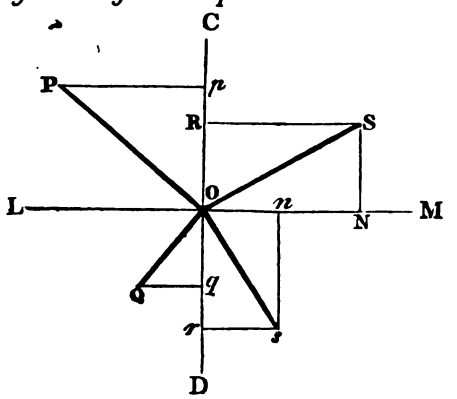
2. For the convenience of calculation, suppose the bodies to be so small, that the lines EAN, FBn (Fig. in p. 184) may be considered as coinciding with LM, and the bodies themselves as concentrated in the point O. In the first instance, let the bodies be *perfectly hard*, and let PO (which represents the velocity of A before impact)$=a$; QO (which represents the velocity of B before impact)$=b$; $\angle POC=\alpha$; $\angle QOD=\beta$; then, by the rules of Trigonometry, (if rad. $=1$) $Pp=a\times\sin.\alpha$; $pO=a\times\cos.\alpha$; $Qq=b\times\sin.\beta$; $qO=b\times\cos.\beta$. Now pO and qO represent the velocities with which the bodies A and B impinge upon each other in opposite directions by *direct impact*, $\therefore$ (by Art. 1. Sect. 19.) their *common velocity* after impact $=\frac{A\times pO-B\times qO}{A+B}=\frac{A\times a\times\cos.\alpha-B\times b\times\cos.\beta}{A+B}$ (X). Hence if we take $OR=X$[a], $ON\,(=Pp)=a\times\sin.\alpha$, $On\,(=Qq)=b$



(a) In this case we estimate the velocity in *A's direction*, and consider $A\times pO$ as *greater than* $B\times qO$; if $A\times pO$ be *less* than $B\times qO$, then OR would lie *above* the plane LM.

$= b \times \sin.\ \beta$, and complete the parallelograms $ONSR$, $OnsR$, the *diagonals* OS, Os of those parallelograms will represent the *directions* and *velocities* of the bodies A and B *after impact.*

3. If the bodies are *perfectly elastic*, then (by Art. 2. Sect. XX.) the *velocity of A after impact*, in direction $CD = \frac{\overline{A-B}.pO - 2B \times qO}{A+B}$; and the *velocity of B after impact* in that *direction* =



$\frac{\overline{A-B}.qO + 2A \times pO}{A+B}$. Take therefore, as before, $ON = a \times \sin.\ \alpha$, $On = b \times \sin.\ \beta$; but (since the bodies in this case have *separate* velocities in *direction* CD) take $OR = \frac{\overline{A-B}.a \times \cos.\ \alpha - 2B \times b \times \cos.\ \beta}{A+B}$;

and $Or = \frac{\overline{A-B}.b \times \cos.\ \beta + 2A \times a \times \cos.\ \alpha}{A+B}$;

complete the parallelograms $ORSN$, $Orsn$, and the diagonals OS, Os will represent the *directions* and *velocities* of A and B *after impact.* In the same

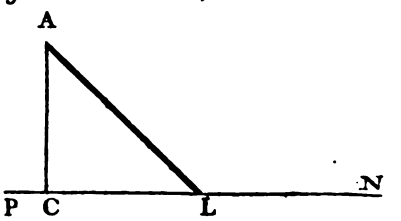
same manner we should proceed if the bodies were *imperfectly elastic*, applying the expressions deduced in Art. 2. Sect. XXI. instead of those just now made use of.

We shall conclude this Lecture with a few observations upon the impact of bodies upon *perfectly hard* planes.

XXIV.

On the Impact of Hard and Elastic Bodies upon perfectly Hard and Immoveable Planes.

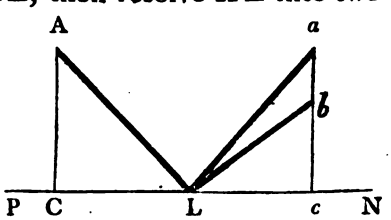
1. If a *perfectly hard* body (A) impinges upon a perfectly hard plane (PN), in a line *perpendicular* to the surface of the plane, then it is evident, that its whole motion will be destroyed by impact, and the body will remain *at rest*. But if, moving with a uniform velocity, it impinges upon the plane in the direction AL *obliquely* to its surface, then resolve AL into two AC, CL, of which AC represents that part of the momentum of the body which is *perpendicular* to PN,



and CL that which is *parallel* to it; the motion represented by AC will be *destroyed* by impact, and the body after impact will proceed along the plane in direction LN, with a momentum proportional to CL; or, since the *velocities* before and

and after impact may also be represented by the lines AL, CL, we have V^y of A *before* impact : V^y *after* impact :: $AL : CL$:: radius : cosine of $\angle ALC$; and to find the velocity *lost* by impact, we have V^y of A *before* impact : the *difference* of the velocities before and after impact (i. e. V^y *lost*) :: $AL : AL - CL$:: radius : radius − cos. $\angle ALC$:: radius : versed sine of the $\angle$ of inclination ALC.

2. If a *perfectly elastic* body impinges *perpendicularly* upon a perfectly hard plane, then, *since the force of restitution is equal to the force of compression*, it will *ascend* from the plane *with the same velocity* as that with which it impinged upon it. But if, moving uniformly, it impinges upon the plane in the *oblique* direction AL, then resolve AL into two AC, CL, of which the *perpendicular* part AC will not be *destroyed* as in the former instance, but will represent *the velocity with which the body ascends from the plane*; and CL will represent the velocity which it has *in direction of the plane*, the same as before. Take therefore $Lc = LC$, and from c draw ca at right angles to Lc and equal to CA, and join La; then La will represent the *direction* and *velocity* of the body after impact. But since Lc, ca are equal to LC, CA, and the angles Lca, LCA are right angles, La will be equal to LA and the angle aLc to the angle ALC; hence the



the body will move *after* impact with the *same velocity* which it had *before* impact, and in a *direction* making the angle of *reflection* aLc equal to the angle of *incidence* ALC.

3. If the body be *imperfectly* elastic, then the velocity with which it begins to ascend from the plane will be diminished in the ratio of *perfect* : *imperfect elasticity* :: 1 : m. Take therefore $cb : ca :: m : 1$, and join Lb; then Lb will represent the *direction* and *velocity* of the body *after impact*. Let $AL = a$, $\angle ALC$ or $aLc = \alpha$, rad. $= 1$, then AC or $ac = a \times \sin.\ \alpha$, and LC or $Lc = a \times \cos.\ \alpha$; now $ac\ (a \times \sin.\ \alpha) : bc :: 1 : m$, $\therefore bc = ma \times \sin.\ \alpha$; hence $Lb^2\ (= Lc^2 + cb^2) = a^2 \times \overline{\cos.\ \alpha}^2 + m^2a^2 \times \overline{\sin.\ \alpha}^2$, and V^y *before* impact : V^y *after* impact $(:: AL : Lb) :: a : a\sqrt{\overline{\cos.\ \alpha}^2 + m^2 \times \overline{\sin.\ \alpha}^2} :: 1 : \sqrt{\overline{\cos.\ \alpha}^2 + m^2 \times \overline{\sin.\ \alpha}^2}$.

4. The foregoing observations apply to bodies moving *uniformly* before and after impact; suppose now an elastic body *to fall by the force of gravity* from any point above a perfectly hard plane; then, if it be *perfectly* elastic, it is evident, that it would rise after impact to the point from which it fell, to acquire the velocity with which it impinged upon the plane; and afterwards *descend again*; having acquired the *same velocity* a second time, it will ascend to the *same height* again after impact, and so it would continue to move on *ad infinitum*. But

if

if it be *imperfectly* elastic, then its velocity (and consequently the *height to which it ascends*) would be *diminished* after each successive impulse upon the plane, and at the end of a certain time its motion would *cease*. In this latter case it will not be difficult to find the space described from the beginning to the end of the motion. For let (a) be the space through which the body falls in the first instance, and $1 : m$ the ratio of *perfect* : *imperfect* elasticity, then, since the velocity is diminished at each successive impulse, in the ratio of $1 : m$, the space described (which varies as the *square* of the velocity) will be diminished in the ratio of $1 : m^2$; i. e. if the space described before the *first* impulse be a, the space described *between the first and second impulse* will be $2m^2a$[a]; the space described *between the second and third impulse* $2m^4a$; &c. &c. so that the *whole space* described before the motion *ceases* will be

$$a + 2m^2a + 2m^4a + \&c. = a + 2m^2a \times \overline{1 + m^2 + m^4 + \&c.}$$

$$= a + \frac{2m^2a}{1-m^2}[b] = \frac{1+m^2}{1-m^2} \times a.$$

5. Let $s = 16\frac{1}{12}$ feet, then (Art. 1. page 33) the *time*

(a) For the space *ascended* through after the first impulse is m^2a; but the body will evidently *descend* through the same space before the second impulse; $\therefore$ *the space described between the first and second impulse* $= m^2a + m^2a = 2m^2a$; and so of the rest.

(b) For $1 + m^2 + m^4 + \&c.$ is an infinite *geometric* series whose *first term* is 1, *common ratio* m^2 (where m is *less than unity*); its *sum* therefore is $\frac{1}{1-m}$.

time of describing $a = \sqrt{\frac{a}{s}}$; of $m^2 a = \sqrt{\frac{m^2 a}{s}} = m\sqrt{\frac{a}{s}}$; $\therefore$ the time *up and down* $m^2 a = 2m\sqrt{\frac{a}{s}}$; for the same reason, the *time up and down* $m^4 a = 2m^2\sqrt{\frac{a}{s}}$; hence the *whole time* of the body's motion $= \sqrt{\frac{a}{s}} + 2m\sqrt{\frac{a}{s}} \times \overline{1 + m + m^2 + \&c.} = \sqrt{\frac{a}{s}} + 2m\sqrt{\frac{a}{s}} \times \frac{1}{1-m} = \overline{1 + \frac{2m}{1-m}} \times \sqrt{\frac{a}{s}} = \frac{1+m}{1-m} \times \sqrt{\frac{a}{s}}$.

EXAMPLE.

A body whose elasticity : perfect elasticity :: 15 : 16, *falls from the height of* 100 *feet upon a perfectly hard plane; it is required to find the whole space* (*S*) *described by the body before its motion ceases; and also the time of its motion* (*T*).

Here $m = \frac{15}{16}$, $a = 100$, $s = 16\frac{1}{12}$ feet $\Bigg\}$ $\therefore S = \frac{1+m^2}{1-m^2} \times a = \frac{48100}{31} = 1551\frac{19}{31}$ feet,

and $T = \frac{1+m}{1-m} \times \sqrt{\frac{a}{s}} =$
$31 \times 2.49 = 77.19$ seconds.

Lecture VI.

ON THE MOTION OF PROJECTILES.

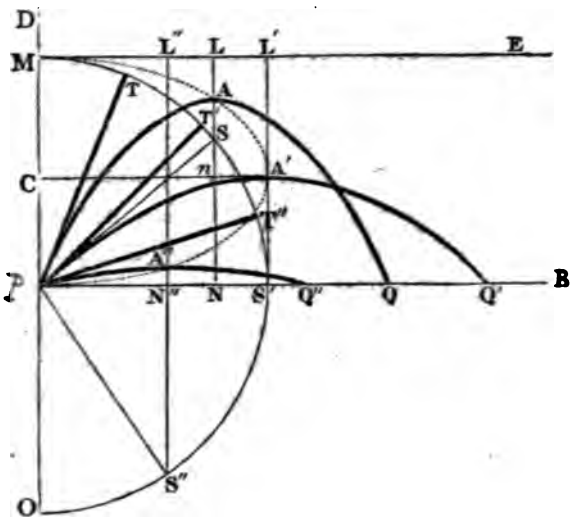
FROM Art. 9, 11. Sect. VIII. it appears, that if a body be projected in any direction not perpendicular to the horizon, it will describe a *parabola*, and that its *velocity* at any point of this parabola will be such as a heavy body would acquire in falling down $\frac{1}{4}$*th part of the parameter* to that point. In Art. 10, of the same Section, we shewed the method of finding the *dimensions* of the parabola thus described; we now proceed to the more general solution of this problem, as applied to bodies projected under all possible circumstances, both with respect to their *velocity* and *direction*.

XXV.

On the method of investigating the dimensions of the Parabola described by a body projected, with a given velocity, at different angles of elevation.

1. Let P be the point from which the body is projected; draw PB *parallel* and PD *perpendicular* to the horizon; take PM equal to the space fallen through to acquire the velocity of projection; then

then (by Art. 11. page 55) PM will be $\frac{1}{4}th$ *part of the parameter* of the parabola at the point P. Produce MP to O, make $PO=MO$, and upon MO describe the semicircle $MS'O$; this semicircle



will be the locus of the *foci*, S, S', S'', &c. of all the parabolas PAQ, $PA'Q'$, $PA''Q''$, &c. which can be described by a body projected (with the velocity acquired in falling down MP) in the directions PT, PT', PT'',[a] &c. Through M draw ME parallel

(a) For take the $\angle^s$ TPS, $T'PS'$, $T''PS''$, &c. equal to MPT, MPT', MPT'', &c. then since PS, PS', PS'', &c. are all equal to PM (i.e. to $\frac{1}{4}$th the parameter to the point P), S, S', S'', &c. will be the foci of all the parabolas PAQ, $PA'Q'$, $PA''Q''$, &c. passing through P, having the same diameter (PO), and the same parameter ($4PM$); and since the $\angle^s$ MPS, MPS', MPS'', &c. are

parallel to PB, and (since $PM=PS=PS'=PS''$, &c.) it will be the *directrix*; through S, S', S'', &c. draw SL, $S'L'$, $S''L''$, &c. parallel to MO, and bisect them in the points A, A', A'', &c. then will A, A', A'', &c. be the *vertices*, and ASN, $A'S'$, $A''S''$, &c. the *axes* of all these Parabolas (See Con. Sect. pages 5, 10, 11.)

2. Through A' draw $A'C$ parallel to ME; then, in tracing the motion of the point S through the semicircle $MS'O'$, it is evident, that whilst the arc MS is *less* than 90°, the vertex of the parabola will be *above* the line CA', and that when it is *greater* than 90°, it will be *below* that line; when that arc (viz. MS') $=90°$, then the vertex will be *in* the line CA'; so that the locus of the *vertices* of all the parabolas PAQ, $PA'Q'$, $PA''Q''$, &c. will lie in a curve extending from M through A' to P, and this curve will be found to be an *ellipse* whose *minor-axis* is PM and *semi-major axis* CA' $(=PM)$ [b].

3. Since

are *double* of the $\angle^s$ MPT, MPT', MPT'', &c. whilst the *latter* pass through all degrees of magnitude between 0 and 90°, the *former* will vary through all degrees of magnitude between 0 and 180°; $\therefore$ all the points S, S' S'', &c. will lie in a *semicircle*; (see page 54, and its notes, where PS is taken equal to $\frac{PN^2}{4PV}=\frac{1}{4}$th the parameter to the point P).

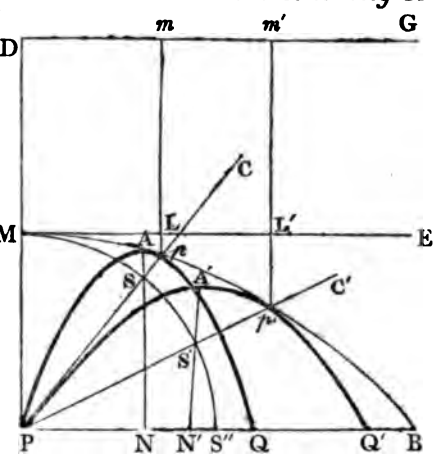
(b) By the property of the parabola $4AL\times AN=PN^2=CA^2$; but $AL=Ln-An$, and $AN=Nn+An=Ln+An$, $\therefore 4AL\times AN=4.\overline{Ln-An}\times\overline{Ln+An}=4Ln^2-4An^2=$ (for $CA'=2Ln$) CA'^2

3. Since the lines PQ, PQ', PQ'', &c. are *bisected* by the axes in the points N, S', N'', &c. (and $\therefore$ are equal to $2PN$, $2PS'$, $2PN''$, &c.) they will *increase* or *decrease* according as the axis of the parabola described *recedes from* or *approaches to* the line MQ; i. e. whilst the vertex of the parabola moves from M to A', the distance at which the projectile strikes the horizontal line PB will keep continually *increasing*; and whilst the vertex moves from A' to P, that distance will keep continually *decreasing*; so that the *greatest* distance at which the body can be made to strike the line PB (with the velocity acquired in falling down MP) will be when the $\angle MPS' = 90^\circ$, in which case the angle of elevation $T'PS'$ $(= \frac{1}{2} MPS') = 45^\circ$, and $PQ' = 2PS' = 2PM = \frac{1}{2}$ the parameter.

4. Produce PM to D (Fig. in p. 197), make $MD = MP$; and with *focus* P, *vertex* M, and *directrix* DG, describe the parabola MpB. Produce PS, PS', &c. to meet the parabola MpB in the points p, p', &c. then by the property of the parabola $Pp = pm$, $Pp' = p'm'$, &c.; but since $PM = MD$, and $PM = PS$, PS', &c. $MD = Lm$,

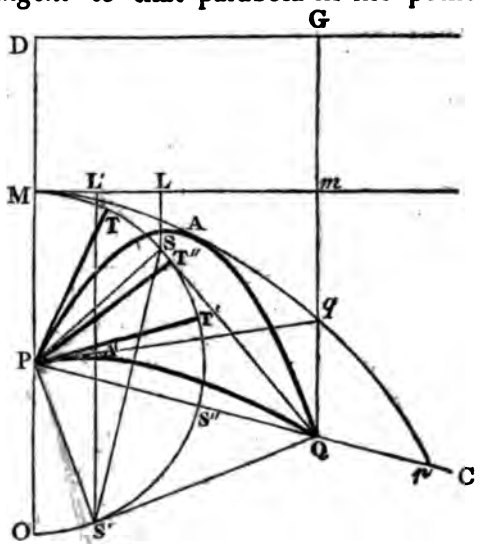
$CA'^2 - 4An^2$. Hence $CA'^2 - 4An^2 = Cn^2$, or $CA'^2 - Cn^2 = 4An^2$, and $CA'^2 - Cn^2 : An^2 :: 4 : 1$; but since $CA' = PM = 2CM$, we have $CA'^2 : CM^2 :: 4 : 1$, $\therefore CA'^2 - Cn^2 : An^2 :: CA'^2 : CM^2$; consequently A is a point in an ellipse whose *center* is C, and CA', CM, the *semi-major* and *semi-minor* axes. (See Cor. to Prop. 7. of the Ellipse.)

$= Lm$, $L'm'$, &c. we have $Pp - PS = pm - Lm$, $Pp' - PS = p'm' - L'm'$, &c.; i. e. $pS = pL$, $p'S' = p'L'$, &c. hence p, p', &c. are *also* points in the parabolas PAQ, $PA'Q'$, &c. Now the lines *which bisect the angles* Ppm, $Pp'm'$, &c. are *tangents* both to the parabola MpB, and to the parabolas PAQ, $PA'Q'$, &c. in the points p, p', &c.; the parabola MpB must consequently *touch* the parabolas PAQ, $PA'Q'$, &c. in those points; these latter parabolas will therefore all lie *within the concavity* of the parabola MpB, as they evidently never *meet it again* after they have passed thro' those points. From hence it appears, that the parabola MpB *limits* the distance to which a body could be projected from the point P (with the velocity acquired in falling down MP) at different angles of elevation; and it is also evident, that the *greatest distance* at which the body could be made to impinge upon the lines PC, PC', &c. drawn in different directions from P, will be in the points p, p', &c. *where that parabola intersects them.*



5. Suppose now PC (Fig. in page 198) to be

be a line drawn, making any angle with PM; describe (as before) the *semicircle* MSO cutting PC in S'', and construct the *limiting parabola* Mqp cutting PC in p, then, from what has just been shewn, S'' will be the *focus* of the parabola[a] described by the body when it impinges upon the line PC at the *greatest possible distance* Pp; and if the arc MS'' be bisected in T'', PT'' will be a *tangent* to that parabola in the point P, and



consequently the *direction* in which the body must be projected. Take any point Q between P and p, and draw QG parallel to PD; with center Q, and radius Qm, describe a circular arc cutting the semicircle MSO in two points S, S', *equally* distant from

(a) To avoid a confusion of lines, this parabola is omitted in the Figure.

from PC[b]; then will $S'S$ be the foci of *two* parabolas PAQ, $PA'Q$ (having the *same parameter* $4PM$, and the *same diameter* PO) which will pass through the point Q.[c] Bisect the angles MPS, MPS' by the lines PT, PT', and they will be *tangents* to the parabolas PAQ, $PA'Q$; if therefore a body be projected in *either* of the directions PT, PT' (with the velocity acquired in falling down MP), it will impinge upon the line PC in the *same point* Q. The directions PT, PT', will moreover be at equal distances *above* and *below* PT''; for the angle $MPS = 2\angle MPT$, and the angle $MPS' = 2\angle MPT'$, $\therefore \angle SPS' = MPS' - MPS = 2 \cdot \overline{MPT' - MPT} = 2TPT'$; hence SPS'' $(= \frac{1}{2} SPS') = TPT'$; take away the common angle SPT', then $T'PS'' = TPS = MPT$; but PT'' bisects the $\angle MPS''$, *it will therefore also bisect the angle* TPT'.

6. From hence we deduce the following method of

(b) That this circular arc will cut the semicircle MSO in *two points equally distant from* PC, is thus proved. Join Pq; and since $qP = qG$, to each add qQ, then $qP + qQ = qG + qQ = QG$; but $qP + qQ$ is greater than PQ, $\therefore QG$ is greater than QP; take away mG $(= PS'')$, then Qm is greater than QS'', consequently a circle described with center Q and radius Qm will cut the semicircle MSO in *two points*, one *above* and the other *below* PS''; and since $PS = PS'$, $QS = QS'$, and PQ *common* to the two triangles PSQ, $PS'Q$, the *angle* SPS'' *will be equal to the angle* $S'PS''$.

(c) For QS, QS', being each equal to Qm, Q is a point both in the parabola whose focus is S, and also in that whose focus is S'; the *directrix* in each case being Mm.

of determining *geometrically* the direction in which a body must be projected from a given point P with the velocity acquired in falling through the perpendicular distance MP, to hit a given mark Q; viz. "*Produce* MP *to* O, *making* $PO = PM$, *and upon* "MO *describe a semicircle; join* PQ, *and let it cut* "*the semicircle in* S''; *through* M *draw* Mm *at* "*right angles to* PM, *and let fall* Qm *perpendi-*"*cularly upon it; with center* Q *and radius* Qm *de-*"*scribe a circular arc, cutting the semicircle* $MS''O$ "*in* S, S'; *join* PS, PS', *and bisect the angles* MPS, "MPS' *by the lines* PT, PT'; a body projected in "either of the directions PT, PT', *will describe a pa-*"*rabola* PAQ *or* $PA'Q$, *which will pass through* Q." If the distance of Q from P be such that $QS'' = Qm$, then Q would be a point in the *limiting parabola* Mqp;(a) and since in this case the circular arc described with center Q and radius Qm would pass through S'', there would be but *one* direction in which the body could be made to impinge upon Q, and that would be found by *bisecting the angle* MPQ. If Qm be *less* than QS'', then the circle described with radius Qm will *not extend* to the semicircle $MS''O$, which indicates that no parabola can be described with parameter $4PM$ which can reach

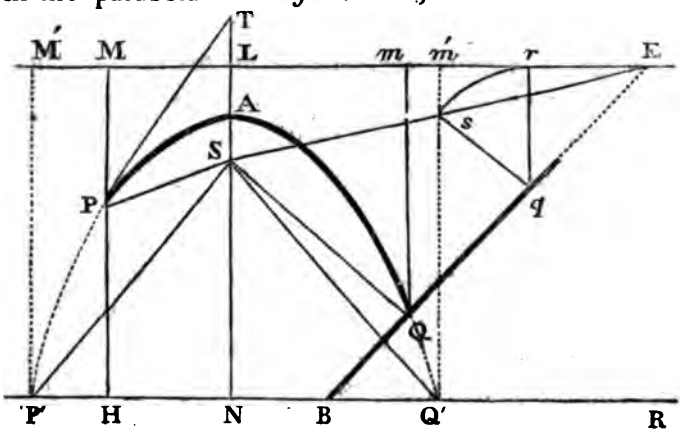
(a) Since $PS'' = mG$, if $Qm = QS''$, then $PS'' + S''Q = Qm + mG$, or $PQ = QG$, $\therefore$ Q is a point in the parabola whose *focus* is P, and *directrix* DG.

reach to Q, or, in other words, that *the velocity of projection is not sufficient to carry a body so far as* Q. Lastly, if PC be a *horizontal* line, then the $\angle T''PS''$ $(=\frac{1}{2}MPS'')=45°$, and since (by Art. 5.) the directions PT, PT' are at equal distances from PT'', it follows, that if the given mark be in the *horizon*, then the two directions in which a body projected with a proper velocity may be made to hit it are *equally distant* from 45°; and if $QS''=Qm$ $(=PM)$, i. e. if $PQ=2PM$[b], (or Q be at a distance from P equal to *twice the space fallen through to acquire the velocity of projection*) then there is only *one* direction in which a body can be made to hit the mark Q with the given velocity, and that is *at an angle of elevation of* 45° *above the horizon.*

7. It only now remains to determine at what point a body projected from the *given point* P (Fig. in p. 202) in the *given direction* PT (with the velocity acquired in falling down MP) will strike the line Bq inclined to the horizon HR in the *given angle* qBR. Produce MP to meet the *horizon* in H, and Bq to meet the *directrix* in E; make the $\angle TPS=$ $\angle MPT$,

(b) When PC becomes a *horizontal* line, $PQmM$ is a *parallelogram*, $\therefore Qm=PM$, and if $QS''=Qm$, then PQ $(=PS''+QS''=PM+Qm)=2PM$; and $2PM$ is the *greatest* distance to which a body can be projected with the velocity acquired in falling down PM (Art. 3.)

$\angle MPT$, and take $PS = PM$, then S will be the *focus* of the parabola described; join SE; in BE take any point q, and draw qr at right angles to ME; with center q and radius qr describe a circular arc cutting SE in s, and join sq; draw SQ parallel to sq, and Q will be the point required. For (Qm being drawn parallel to qr) we have, by similar triangles, $SQ : sq$ ($:: EQ : Eq$) $:: Qm : qr$; but $sq = qr$, $\therefore SQ = Qm$, and consequently Q is a point in the parabola whose *focus* is S, and *directrix* ME.



To determine the point where the body would strike the *horizon*; with center S and radius $= MP + PH$ (i. e. MH) describe a circular arc cutting HR in Q'; then, since $SQ' = MH = Q'm'$ ($Q'm'$ being drawn parallel to HM), Q' is a point in the parabola; and if the *point of projection* (P') be also in the horizon, then it is only necessary to describe a circular arc with center S and radius SP', for in this case $SQ' = SP' = P'M' = Q'm'$, and $\therefore Q'$ a point in the parabola as before.

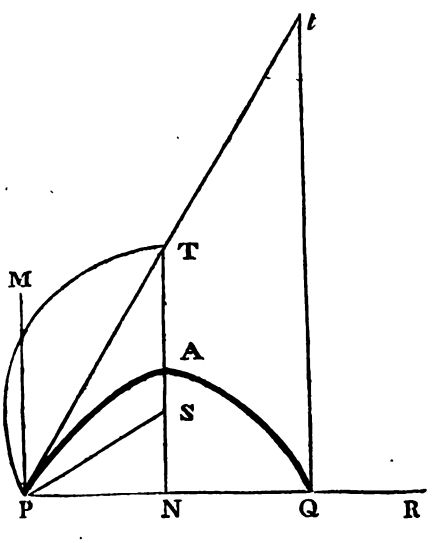
Having

Having thus explained the geometrical principles upon which problems relating to the motion of projectiles are constructed, we now proceed to consider the different circumstances under which their motion is performed; beginning with the case of bodies projected from one point to another *along the horizon*.

XXVI.

On *the Method of ascertaining the* Range[a], Greatest Altitude, Time of Flight, &c. *of Bodies projected along the Horizon.*

1. Let PR represent the *horizon*; suppose a body to be projected in the direction PT with the velocity acquired in falling down MP, and let PAQ be the parabola described; make the $\angle TPS = \angle MPT$, and take $PS = PM$, then S will be the *focus*; through S draw TN at right $\angle$s to PQ; then PQ will be



(a) The *range* is the distance of the point from which the body is projected to the point where it descends again upon the horizon.

be bisected by the *axis* in N, and the *vertex* of the parabola will bisect the *subtangent* TN in A. Draw Qt at right $\angle^{r}$ to PR, and produce PT to meet it in t; then PQ is the *range* of the projectile, AN its *greatest altitude* above the horizon, and the *time of flight* will be the time of describing Pt with the velocity of projection continued uniformly, or the time of descending from rest through the space tQ by force of gravity.

2. With center S and *radius* SP or ST (for $ST = SP$ by Prop. 3 of Parabola) describe a circular arc PT; then PN is the *sine*, TN the *versed sine*, and PT the *chord* of the angle PST to that radius. Now let the angle of projection $TPN = \alpha$, then the $\angle MPT = 90^{\circ} - \alpha$, and $\therefore$ the $\angle MPS\ (= 2MPT) = 180^{\circ} - 2\alpha$; but since TN is parallel to MP, the $\angle PSN = \angle MPS$, $\therefore PSN = 180^{\circ} - 2\alpha =$ *supplement* of 2α; hence the $\angle PST$ (of which PSN is *also* the supplement) must be equal to 2α; and PT (which is the *chord* of 2α) must be equal to $2 \sin. \alpha$. (See Pl. Trig. page 9.) Let PM or $PS\ (= \frac{1}{4}$ parameter to the point $P) = p$; then we have,

$PS\,(p) : PN :: \text{rad} : (1) : \text{sin. } \angle PSN \text{ or } PST\ (\text{sin. } 2\alpha)$
$\therefore PN = p \times \text{sin. } 2\alpha.$

$PS\,(p) : TN :: \text{rad} : (1) : \text{ver. sin. } PST\ (\text{ver. sin. } 2\alpha),$
$\therefore TN = p \times \text{ver. sin. } 2\alpha.$

$PS\,(p) : PT :: \text{rad} : (1) : \text{chord of } PST\ (2 \text{ sin. } \alpha),$
$\therefore PT = 2p \times \text{sin. } \alpha.$

3. But the *range* $PQ = 2PN = 2p \times \text{sin. } 2\alpha$; the *greatest*

greatest altitude $AN=\frac{1}{2}TN=\frac{1}{2}p\times$ ver. sin. 2α; and by sim. $\triangle^{s}\ Pt : PT\ (2p\times \text{sin. } \alpha) :: PQ : PN :: 2 : 1$, $\therefore Pt=4p\times \text{sin. } \alpha$; hence the *time of flight* =the time of describing (Pt, or) $4p\times$sin. α with the uniform velocity of projection. Let this velocity of projection $=v$, and $m=16\frac{1}{12}$ feet, then (by Art. 1. page 33.) $p=\frac{v^2}{4m}$. Substituting this value for p in the foregoing expressions, and putting R for the *range*, A for the *greatest altitude*, and T for the *time of flight*, we have the following equations and variations; viz.

$$R=2p\times \text{ sin. } 2\alpha^{(a)}=\frac{v^2\times \text{sin. } 2\alpha}{2m}\propto v^2\times \text{sin. } 2\alpha;$$

$$A=\tfrac{1}{2}p\times \text{ver. sin. } 2\alpha=\frac{v^2\times \text{ver. sin. } 2\alpha}{8m}\propto v^2\times \text{ver. sin. } 2\alpha;$$

$$T=\frac{4p\times \text{ sin. } \alpha}{v}{}^{(b)}=\frac{v\times \text{sin. } \alpha}{m}\propto v\times \text{sin. } \alpha;$$

from which the relation of these several quantities may in all cases be determined.

4. If v be given, then $R\propto$ sin. 2α, and consequently will be *greatest* when $\alpha=45^{\circ}$, in which case

(a) If $\alpha=15^{\circ}$, or 75°, then sin. $2\alpha=$sin. $30^{\circ}=\frac{1}{2}$ rad.$=\frac{1}{2}$, and $R=2p\times\frac{1}{2}=p$; hence if a body be projected in an $\angle$ of 15°, or 75°, its range upon the horizon is just equal to the space fallen through to acquire the velocity of projection.

(b) For in general, $T=\frac{space}{velocity}$; $\therefore$ the time of describing the space ($4p\times$sin. α) with the uniform velocity (v)$=\frac{4p\times \text{sin. } \alpha}{v}$.

case $R\,(=2p\times\sin.\,2\alpha=2p\times\text{rad.})=2p=\frac{1}{2}$ the parameter; and $A=\frac{1}{2}\,p\times\text{ver. sin. }2\alpha=\frac{1}{2}\,p\times\text{rad.}=\frac{1}{2}\,p$. $A\propto\text{ver. sin. }2\alpha$, and $\therefore$ is *greatest* when $\alpha=90^{\circ}$, in which case $A=(\frac{1}{2}\,p\times\text{ver. sin. }2\alpha=\frac{1}{2}\,p\times\text{diam.}-)$ $p=\frac{1}{4}$ parameter. $T\propto\sin.\,\alpha$, and $\therefore$ is *greatest* when $\alpha=90^{\circ}$, in which case $T=\left(\frac{4p\times\sin.\,\alpha}{v}=\frac{4p}{v}=\right)$ *time down the parameter*.[a] If the angle of projection (α) be given, then R and A vary as v^2, and T varies as v.

5. Since $\sin.\,2\alpha=2\sin.\,\alpha\times\cos.\,\alpha$ (Trig. page 28), and $\text{ver. sin. }2\alpha=2\,.\,\overline{\sin.\,\alpha}^{\,2}$[b]; we have $R=4p\times\sin.\,\alpha\times\cos.\,\alpha$, and $A=p\times\overline{\sin.\,\alpha}^{\,2}$; hence $A\times R=4p^2\times\overline{\sin.\,\alpha}^{\,3}\times\cos.\,\alpha$. Now the *area* of the parabola described $=\frac{2}{3}A\times R$; this area therefore is *equal* to $\frac{8p^2\times\overline{\sin.\,\alpha}^{\,3}\times\cos.\,\alpha}{3}$, and (when p is given) *varies* as $\overline{\sin.\,\alpha}^{\,3}\times\cos.\,\alpha$.[c]

EXAM.

(a) Let t=time down p, then the time of describing $4p$ with the uniform velocity $(v)=2t$, $\therefore T=2t$; but the time *down the parameter* : time down p (t) :: $\sqrt{4p}:\sqrt{p}::2:1$, $\therefore$ *the time down the parameter*$=2t=T$.

(b) For (Trig. page 9.) $\text{ver. sin. }2\alpha=\frac{\overline{\text{chord of }2\alpha}^{\,2}}{\text{diam.}}=\frac{\overline{4\sin.\,\alpha}^{\,2}}{2}$ $=2.\overline{\sin.\,\alpha}^{\,2}$.

(c) Let $x=\sin.\,\alpha$, then $\cos.\,\alpha=\sqrt{1-x^2}$, $\therefore$ when this area is *greatest*, $x^3\times\sqrt{1-x^2}=\text{max.}$; or $x^6\,.\,\overline{1-x^2}=\text{max.}$; hence $x^6-x^8=\text{max.}$ and $6x^5\dot{x}-8x^7\dot{x}=0$, $\therefore x^2=\frac{3}{4}$, and $x=\sqrt{\frac{3}{4}}=\sin.\,60^{\circ}$; a body therefore projected with a *given velocity* describes a parabola of the *greatest area*, when the angle of projection is 60°.

EXAM. 1.

A body is projected in a direction making an angle of 15° *with the horizon, with a velocity of* 40 *feet in a second; Find its* range, greatest altitude, *and* time *of flight.*

Here $\alpha = 15°$, $v = 40$ feet, $m = 16\frac{1}{12}$. By Art. 3, $R = \frac{v^2 \times \text{sin. } 2\alpha}{2m} = \frac{1600 \times \text{sin. } 30°}{32\frac{1}{6}} = \frac{1600 \times \frac{1}{2}}{32\frac{1}{6}} =$

$\frac{4800}{193} = 24.87$ feet.

Also, $A = \frac{v^2 \times \text{ver. sin. } 2\alpha}{8m} = \frac{1600 \times \text{ver. sin. } 30°}{128\frac{2}{3}} =$

$\frac{1600 \times .134}{128\frac{2}{3}}$ [d] $= \frac{4800 \times .134}{386} = 1.66$ feet.

And $T = \frac{v \times \text{sin. } \alpha}{m} = \frac{40 \times .258}{16\frac{1}{12}}$ [e] $= \frac{480 \times .258}{193} =$.641″.

Hence its horizontal range [f] is 24.87 feet, its greatest

([d]) Ver. sin. 30° = rad. − cos. 30° = 1 − .866, &c. = .134. See Plane Trig. page 16.

([e]) Sin. 15° = .2588190. See Plane Trig. page 17.

([f]) By Art. 6, page 201, a body projected with a given velocity at angles equally distant from 45° will describe the *same horizontal range*; in order, therefore, that the equation $R = \frac{v^2 \times \text{sin. } 2\alpha}{2m}$ may be general, it is necessary to shew that sin. 2α *is the same* in *both cases*. Let $\alpha = 45° \pm d$, then $2\alpha = 90° \pm 2d$; but sin. $90° + 2d$ = sin. $90° - 2d$, ∴ sin. 2α is the *same*, whether $\alpha = 45° + d$ or $45° - d$.

greatest altitude above the horizon 1.66 feet, and its time of flight .641 of a second.

EXAM. 2.

A body is projected in an angle of 60°, *and descends to the horizon at a distance of* 100 *feet from the point of projection*; *With what* velocity *was it projected, and what was its* greatest altitude *and* time of flight?

Here $R=100$, $\alpha=60^\circ$, $m=16\frac{1}{12}$. Since $R=\frac{v^2\times\sin.\,2\alpha}{2m}$, $\therefore v^2=\frac{2mR}{\sin.\,2\alpha}=\frac{32\frac{1}{6}\times 100}{\sin.\,120^\circ}=\frac{32\frac{1}{6}\times 100}{\frac{\sqrt{3}}{2}}$ (a)

$=\frac{19300}{3\sqrt{3}}=3699.42$; hence $v=\sqrt{3699.42}=60.8$ feet in 1″.

Also $A=\frac{v^2\times\text{ver. sin. }2\alpha}{8m}=\frac{3699.42\times\text{ver. sin. }120^\circ}{128\frac{2}{3}}$

$=\frac{3699.42\times\frac{3}{2}}{128\frac{2}{3}}$ (b) $=43.1$ feet.

And $T=\frac{v\times\sin.\,\alpha}{m}=\frac{60.8\times\sin.\,60^\circ}{16\frac{1}{12}}=\frac{60.8\times\frac{\sqrt{3}}{2}}{16\frac{1}{12}}$

$=\frac{364.8\times\sqrt{3}}{193}=3.26''$.

EXAM. 3.

Given the horizontal range *and* time of flight *of a projectile, it is required to find the* angle *in which it*

(a) Sin. $120^\circ=$sin. $60^\circ=\sqrt{\frac{3}{4}}$. Trig. page 13.

(b) Ver. sin. $120^\circ=$rad. + cos. $60^\circ=$rad. + sin. $30^\circ=1+\frac{1}{2}=\frac{3}{2}$

it was projected, the velocity *of projection, and its* greatest altitude *above the horizon.*

Since $R = \frac{v^2 \times \text{sin. } 2\alpha}{2m}$,

$\therefore\ v^2 = \frac{2mR}{\text{sin. } 2\alpha} = \frac{2mR}{2 \times \text{sin. } \alpha \times \text{cos. } \alpha} = \frac{mR}{\text{sin. } \alpha \times \text{cos. } \alpha}$;

also $T = \frac{v \times \text{sin. } \alpha}{m}$;

$\therefore\ v = \frac{mT}{\text{sin. } \alpha}$, and $v^2 = \frac{m^2T^2}{\text{sin. } \alpha^2}$. Hence

$\frac{m^2T^2}{\text{sin. } \alpha^2} = \frac{mR}{\text{sin. } \alpha \times \text{cos. } \alpha}$, or

$\frac{mT^2}{\text{sin. } \alpha} = \frac{R}{\text{cos. } \alpha}$; $\therefore$

$\frac{m^2T^4}{\text{sin. } \alpha^2} = \frac{R^2}{\text{cos. } \alpha^2} = \frac{R^2}{1 - \text{sin. } \alpha^2}$, which reduced, gives

$$\text{sin. } \alpha = \frac{mT^2}{\sqrt{R^2 + m^2T^4}}.$$

Hence α is known, and consequently $v \left(= \frac{mT}{\text{sin. } \alpha} \right)$ and $A \left(= \frac{v^2 \times \text{ver. sin. } 2\alpha}{8m} \right)$ may be found.

This is given merely as a specimen of examples of this kind; for by comparing the expressions in Art. 3, page 205, it is evident, that of the five quantities R, A, T, α, v, any *two* being given, the *other three* may always be found.

EXAM.

EXAM. 4.

Given the proportion between the range *and* altitude *of a body projected with a* given velocity; *It is required to find the* angle *of projection.*

By Art. 5, page 206.

$$R : A :: 4p \times \sin.\alpha \times \cos.\alpha : p \times \overline{\sin.\alpha}^2$$
$$:: 4\cos.\alpha : \sin.\alpha;$$
$$\therefore R^2 : A^2 :: 16\,\overline{\cos.\alpha}^2 : \overline{\sin.\alpha}^2 :: 16.\overline{1-\overline{\sin.\alpha}^2} : \overline{\sin.\alpha}^2,$$
and $R^2 : 16A^2 \ldots\ldots :: 16.\overline{1-\sin.\alpha}^2 : 16\,\overline{\sin.\alpha}^2.$
Hence $R^2 + 16A^2 : 16A^2 :: 16 : 16.\overline{\sin.\alpha}^2 :: 1 : \overline{\sin.\alpha}^2,$

$$\therefore \sin.\alpha = \frac{4A}{\sqrt{R^2+16A^2}}.$$

Cor. If $R = A$, then $\sin.\alpha = \frac{4}{\sqrt{17}} = .9701 =$ sin. 75° 58′, i. e. "in order that the *greatest* "*altitude* of a projectile may be equal to its *range*, "its direction must make an angle of about 76° "with the horizon."

EXAM. 5.

Find the velocity *and* time of flight *of a body projected from one extremity of the base of an equilateral triangle, and in the direction of the side adjacent to that extremity, towards an object placed in the other extremity of the base.*

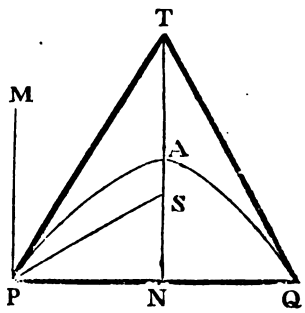
Let S = *side of the triangle*; then by Exam. 2, $v^2 = \frac{2mR}{\sin.2\alpha} =$ (in this case) $\frac{2mS}{\sin.120^\circ}$ (= for sin. 120° =

sin.

$$\text{sin. } 60^\circ = \frac{\sqrt{3}}{2}\Big) \frac{4mS}{\sqrt{3}}, \quad \therefore v = \frac{2\sqrt{mS}}{\sqrt[4]{3}} \text{ feet in } 1''$$

$$T = \frac{v \times \text{sin. } \alpha}{m} = \frac{2\sqrt{mS}}{\sqrt[4]{3}} \times \frac{\sqrt{3}}{2m} = \frac{\sqrt{S}}{\sqrt{3}m} \text{ seconds.}$$

Or thus *geometrically*. Let TPQ be the triangle, MP the space fallen through to acquire the velocity of projection; make $\angle TPS = \angle MPT$, and take $PS = PM$, then S will be the *focus*; join TS and produce it to N, and TN will be the *axis*; Now $PN = \frac{1}{2} PQ$, and since $\angle SPN = 30^\circ$, $SN = \frac{1}{2} SP$; hence $SP^2 (= PN^2 + SN^2) = \frac{1}{4} PQ^2 + \frac{1}{4} SP^2$, and $\frac{3}{4} SP^2 = \frac{1}{4} PQ^2 = \frac{1}{4} S^2$, or $SP = \frac{S}{\sqrt{3}}$; $\therefore PM = \frac{S}{\sqrt{3}}$, and $v = (\sqrt{4m \times PM} =) \sqrt{\frac{4mS}{\sqrt{3}}} = \frac{2\sqrt{mS}}{\sqrt[4]{3}}$, as before.



EXAM. 6.

Compare the areas of the parabolas described by two bodies projected together from the same point in the horizon, and with the same velocity, the angles of projection being to each other :: 2 : 1.

By Art. 5. the area of the parabolas described by a body projected with the *same* velocity at *different* angles of elevation, vary as $\overline{sine}^3 \times cosine$ of that angle.

angle. Let the angles described in the present instance be to each other :: $2\beta : \beta$; then

Area des^{d} at $\angle 2\beta$: area des^{d} at $\angle \beta$:: $\overline{\sin. 2\beta}|^{3} \times \cos. 2\beta : \overline{\sin. \beta}|^{3} \times \cos. \beta :: 8 \times \overline{\sin. \beta}|^{3} \times \overline{\cos. \beta}|^{3} \times \overline{2\overline{\cos. \beta}|^{2} - 1}$ [a] : $\overline{\sin. \beta}|^{3} \times \cos. \beta :: 8\overline{\cos. \beta}|^{4} \times \overline{2\overline{\cos. \beta}|^{2} - 1} : 1 :: 16\overline{\cos. \beta}|^{4} - 8\overline{\cos. \beta}|^{2} : 1$.

1. Let $\beta = 30^{\circ}$, then cos. $\beta = \sin. 60^{\circ} = \frac{\sqrt{3}}{2}$; $\therefore$ the area des^{d} at an $\angle$ of 60° : area des^{d} at an $\angle$ of $30^{\circ} :: 16 \times \frac{9}{16} - 8 \times \frac{3}{4} : 1 :: 9 - 6 : 1 :: 3 : 1$. The same conclusion might indeed be immediately deduced from the expression for the *altitude*; for this is the particular case when two bodies projected at angles of elevation, which are to each other :: 2 : 1, are also projected in directions *equally distant from* 45°; the *ranges* described being therefore *equal*, the *areas* described will be to each other as the *altitudes*[b] of the parabolas; but the *altitude* $\propto$ ver. sin. 2α, $\therefore$ area des^{d} at $\angle 60^{\circ}$: area des^{d} at $\angle 30^{\circ}$:: ver. sin. 120° : ver. sin. $60^{\circ} :: \frac{3}{2}$ rad. : $\frac{1}{2}$ rad. :: 3 : 1.

2. Suppose it were required to find two such angles of elevation as should make the areas *equal*.

([a]) For sin. $2\beta = 2$ sin. $\beta \times$ cos. β, and cos. $2\beta = 2\overline{\cos. \beta}|^{2} - 1$, see Plane Trigonometry, page 28.

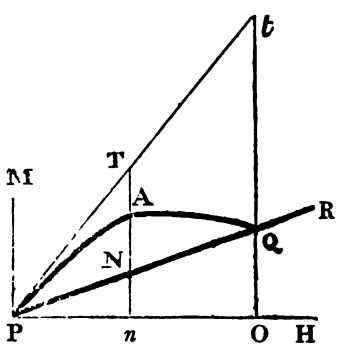
([b]) By Art. 5, the area $\propto A \times R$; if R be *given*, then area $\propto A$.

equal. Let x = cosine of the lesser angle, then the proportion of the areas becomes $16x^4 - 8x^2 : 1$; in this case therefore $16x^4 - 8x^2 = 1$, or $x^4 - \frac{1}{2}x^2 = \frac{1}{16}$; which equation solved gives $x^2 = \frac{1+\sqrt{2}}{4} = \frac{2.4142}{4} = .6035$, $\therefore x = \sqrt{.6035} = .7762$ = cos. 39° 4′; hence if two bodies are projected from the same point and with the same velocity, in directions making $\angle^s$ of 78° 8′ and 39° 4′ with the horizon, the *areas* of the parabolas described will be *equal* to each other.

XXVII.

On the Method of finding the Range, Altitude, and Time of Flight, of Bodies projected along Planes inclined to the Horizon.

1. Let the body be projected from the point P, in direction Pt; and let MP be the space through which the body falls to acquire the velocity of projection, POH the *horizon*, PQR the *plane along which the body is projected*, and PAQ the parabola described. Thro' Q draw tQO perpendicular to the horizon; then the *range* upon the plane is PQ, and the *time of flight* will be the time of describing Pt with the uniform velocity of projection as



as before; moreover, if PQ be bisected in N, and TN be drawn parallel to tQ, then (by Conics, Art. 2, page 127) AN will be the *greatest altitude* of the projectile above the plane PR.

2. Let the angle of *elevation* of the plane above the horizon $(QPO)=\beta$; the angle of *projection* $(tPQ)=\alpha$; then the angle MPt or its *equal* $PtQ=90^\circ-\overline{\alpha+\beta}$, $\therefore$ sin. $PtQ=$ cos. $\overline{\alpha+\beta}$; the $\angle PQO=$ complement of QPO, $\therefore$ sin. PQO or its *equal* sin. $PQt=$ cos. β. Let $PM=p$ as before; then, by the property of the Parabola, $4PM\times tQ$, or $4p\times tQ=Pt^2$ [a]; hence we have

$4p : Pt :: Pt : tQ ::$ sin. PQt (cos. β) : sin. tPQ (sin. α) [X],

and $Pt \cdot PQ \ . \ . \ . \ . ::$ sin. PQt (cos. β) : sin. PtQ (cos. $\overline{\alpha+\beta}$);

ex æquo, $4p : PQ \ . \ . \ . \ . :: \overline{\text{cos. } \beta}^2 \ . \ . : \text{sin. } \alpha \times \text{cos. } \overline{\alpha+\beta}$ [Y].

[b] From proportion [X] $4p : tQ :: \overline{\text{cos. } \beta}^2 : \overline{\text{sin. } \alpha}^2$ [Z].

Hence from proportion [Y] we have $PQ=\dfrac{4p\times \text{sin. } \alpha\times \text{cos. } \overline{\alpha+\beta}}{\overline{\text{cos. } \beta}^2}$,

. [Z] . . . $tQ=\dfrac{4p\times \overline{\text{sin. } \alpha}^2}{\overline{\text{cos. } \beta}^2}$,

. [X] . . . $Pt=\dfrac{4p\times \text{sin. } \alpha}{\text{cos. } \beta}$.

3. Let

(a) For (See Note c and Fig. 1. page 54.) $4SP\times PV=QV^2$, where $PV=NQ$, and $PN=QV$, $\therefore 4SP\times NQ=PN^2$; but $PM=SP$, and Pt, tQ of *this* figure correspond to PN and NQ in *Figure, page 54*, $\therefore 4PM\times tQ=Pt^2$.

(b) For when *three* quantities are in *continued* proportion, the first is to the third in the *duplicate* ratio of the *first to the second*, or of the *second to the third*; $\therefore 4p : tQ :: Pt^2 : tQ^2 :: \overline{\text{cos. } \beta}^2 : \overline{\text{sin. } \alpha}^2$.

3. Let now v = velocity of projection, then $p = \frac{v^2}{4R}$, and $4p = \frac{v^2}{m}$ as before; substituting this value for $4p$ in the foregoing expressions, we have the following equations and variations; viz.

(*R*) The *range* $(PQ) = \frac{v^2 \times \text{sin.}\ \alpha \times \text{cos.}\ \overline{\alpha+\beta}}{m\,.\,\text{cos.}\ \beta\,|^2} \propto \frac{v^2 \times \text{sin.}\ \alpha \times \text{cos.}\ \overline{\alpha+\beta}}{\text{cos.}\ \beta\,|^2}$ (a)

(*A*) *Altitude* $(AN) = \frac{1}{4}tQ$ (b) $= \frac{v^2 \times \overline{\text{sin.}\ \alpha}\,|^2}{4m\,.\,\text{cos.}\ \beta\,|^2}$ (c) $\propto \frac{v^2 \times \overline{\text{sin.}\ \alpha}\,|^2}{\text{cos.}\ \beta\,|^2}$.

(*T*) *Time of flight* $= \frac{Pt}{v} = \frac{v \times \text{sin.}\ \alpha}{m \times \text{cos.}\ \beta} \propto \frac{\text{sin.}\ \alpha}{\text{cos.}\ \beta}$.

(a) Suppose v and β to be given, then $R \propto \text{sin.}\ \alpha \times \text{cos.}\ \overline{\alpha+\beta}$; and if R and β be given, $v^2 \propto \frac{1}{\text{sin.}\ \alpha \times \text{cos.}\ \overline{\alpha+\beta}} \propto \frac{1}{R}$. But if v and β be given, R is a *maximum* when PT bisects the angle MPQ (by Art. 5, page 198); and since in this case $v^2 \propto \frac{1}{R}$, *it* will be a *minimum*; hence, whether it be required to find the *greatest range* which a body would describe with a *given velocity* upon an inclined plane of given elevation, or the *least velocity* with which it would describe a *given range* upon the same plane, and whether the body be projected *upwards* or *downwards* upon the plane (for, in the Fig. of page 198, the line PC may be of *any* given inclination), we must in each case make the *angle of projection*, such that it *bisects* the angle MPQ.

(b) By Prop. 10th of the parabola, $AN = \frac{1}{2}TN$; and by sim. Δ^s $tQ : TN :: PQ : PN$; but $PN = \frac{1}{2}PQ$, $\therefore TN = \frac{1}{2}tQ$, and $AN = \frac{1}{2}TN = \frac{1}{4}tQ$.

(c) If it were required to find the greatest altitude (An) above

EXAM. I.

A body is projected up a plane inclined to the horizon in an angle of 30°, *with a velocity of* 100 *feet in a second; its direction makes an angle of* 45° *with the plane; What is its range upon the plane, its greatest altitude above the plane, and its time of flight?*

Here $v = 100$ feet, $\beta = 30°$, $\alpha = 45°$, $m = 16\frac{1}{12}$ feet

$$\therefore R = \frac{v^2 \times \sin. \alpha \times \cos. \alpha + \beta}{m . \overline{\cos. \beta}^2} = \frac{10000 \times \sin. 45° \times \cos. 75}{16\frac{1}{12} \times \overline{\cos. 30°}^2} =$$

$$\frac{10000 \times \frac{1}{\sqrt{2}} \times \cos. 75°}{16\frac{1}{12} \times \frac{3}{4}} = \frac{10000 \times .7071 \times .2588}{12.06} =$$

151.73 feet. Also

$$A = \frac{v^2 \times \overline{\sin. \alpha}^2}{4m . \overline{\cos. \beta}^2} = \frac{10000 \times \frac{1}{2}}{64\frac{1}{3} \times \frac{3}{4}} = \frac{5000}{48.25} = 103.62 \text{ feet.}$$

And

$$T = \frac{v \times \sin. \alpha}{m \times \cos. \beta} = \frac{100 \times \frac{1}{\sqrt{2}}}{16\frac{1}{12} \sqrt{\frac{3}{2}}} = \frac{100 \times .7071}{13.91} = 5.08 \text{ seconds.}$$

the *horizon*, then Nn must be added to it; and Nn may always be found when the *range* is known, for PQ (R) : QO :: rad. : sin.$\angle QPO$:: 1 : sin. β, $\therefore QO = R \times \sin. \beta$; and since $PN = \frac{1}{2} PQ$, $Nn = \frac{1}{2} QO = \frac{R \times \sin. \beta}{2}$.

EXAM. 2.

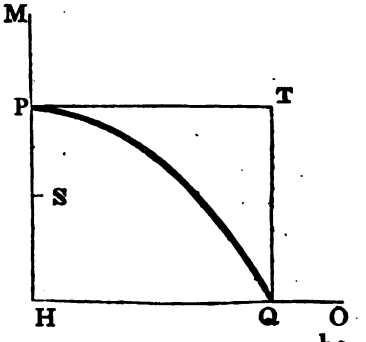
A body projected in a direction making an angle of 30° with a plane whose elevation above the horizon was also 30°, fell upon the plane at the distance of 200 feet from the point of projection; With what velocity was it projected?

Here $R=200$ feet, $\beta=30^\circ$, $\alpha=30^\circ$, $m=16\frac{1}{12}$. Now $R=\frac{v^2\times\sin.\alpha\times\cos.\overline{\alpha+\beta}}{m.\overline{\cos.\beta}^2}$, $\therefore v^2=\frac{m.\overline{\cos.\beta}^2\times R}{\sin.\alpha\times\cos.\overline{\alpha+\beta}}=$ (in this case) $\frac{m\times\overline{\cos.30^\circ}^2\times R}{\sin.30^\circ\times\cos.60^\circ}=\frac{16\frac{1}{12}\times\frac{3}{4}\times200}{\frac{1}{2}\times\frac{1}{2}}=\frac{119400}{12}=9950$, $\therefore v=\sqrt{9950}=91.02$ feet in a second.

XXVIII.

On the Method of determining the Range, Altitude, and Time of Flight of a Body projected from any given point, *to the* Horizon, *or to* a plane inclined in any given angle *to the horizon.*

1. Let a body be projected from the given point *P*, with the velocity acquired in falling down *MP*, and in a direction *PT parallel to the horizon HQO*. Let *P* be the parabola described; then since *PT* is at right angles to *PH*, *P* will be

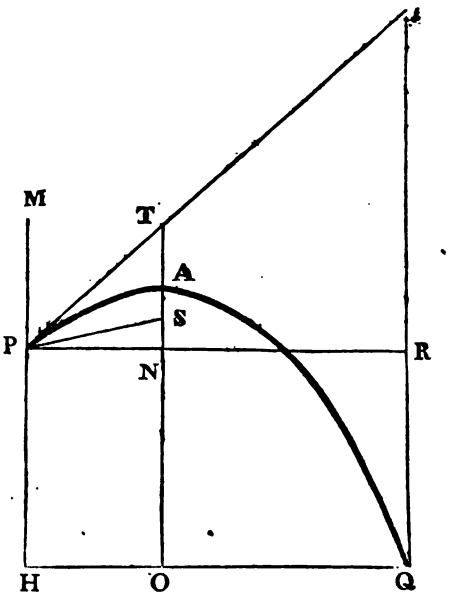


be the *vertex*; and if PS be taken equal to PM, S will be the *focus*. By the property of the parabola $HQ^2 = 4SP \times PH = 4PM \times PH$, $\therefore$ the *range* $HQ = \sqrt{4PM \times PH}$. If $PM = PH$, then $HQ = \sqrt{4PH^2} = 2PH$: i. e. "If a body be projected from "any point P with the velocity acquired in falling "down PH (the perpendicular distance of P from "the horizon); then the *range* described upon "the horizon is equal to *twice* that perpendicular "distance."

2. Let $PM = p$, v = velocity of projection, PH (the perpendicular distance of the point P from the horizon) $= h$, $m = 16\frac{1}{12}$ feet, then $p = \frac{v^2}{4m}$, $\therefore$ the *range* $HQ\,(R) = \sqrt{4ph} = \sqrt{\frac{v^2 \times h}{m}}$, and $R^2 = \frac{v^2 \times h}{m}$; hence $v^2 = \frac{mR^2}{h}$, and $v = R\sqrt{\frac{m}{h}}$; from which, when R and h are given, v may be determined. Complete the parallelogram $PHQT$, then the *time of flight* T is equal to the time of falling down TQ or PH (h), $\therefore$ by Art. 1. page 33) $T = \sqrt{\frac{h}{m}}$.

3. Suppose now the body to be projected in the direction Pt, making a *given angle* tPR, with the line PA drawn parallel to the horizon. Let PAQ be the parabola described; make $\angle TPS = \angle MPT$, and

and $PS = PM$; through S draw TSO perpendicular to the horizon, then A is the *vertex*, S the *focus*, and AO the *axis* of the parabola. The *range* $HQ = HO + OQ = PN + OQ =$ (for $OQ^2 = 4AS \times AO$) $PN + \sqrt{4AS \times AO} = PN + \sqrt{4AS.AN + PH}$, for $AO = AN + NO = AN + PH$, where AO is also the *greatest altitude*; and if tQ be drawn parallel to AO, then the *time of flight* = time of falling down tQ, or $tR + RQ$, or $tR + PH$. $= \sqrt{\frac{tR + PH}{m}}$.



4. Let $\angle tPR = \alpha$, then (by Art. 2. page 204) $PN = p \times \sin. 2\alpha$; and (by Art. 5. page 206) $AN = p \times \sin.\ \alpha^2$. Now, by the property of the parabola, $SP = AN + AS$, $\therefore AS = SP - AN =$ (for $SP = PM = p$) $p - p \times \sin.\ \alpha^2 = p.\overline{1 - \sin.\ \alpha}^2 = p \times \cos.\ \alpha^2$; also, since $PR\ (= HQ) : tR ::$ rad. : tan. $\angle tPR ::$ 1 : tan. α, $tR = HQ \times$ tan. α. Substituting these values for PN, AN, AS and tR in the foregoing expressions, and putting R, A, T, h for the *range*, *altitude*,

[illegible] time of flight and height of P above the [illegible] we have

$$R = 2 \times \sin. z \cos. z - \sqrt{4p \cdot \overline{\cos. z}^2 \cdot p \times \overline{\sin. z}^2 - h} \quad (B)$$

$$A = p \times \overline{\sin. z}^2 - h$$

$$T = \sqrt{\frac{2R - PH}{m}} = \sqrt{\frac{HQ \times \tan. z - h}{m}} \quad (C).$$

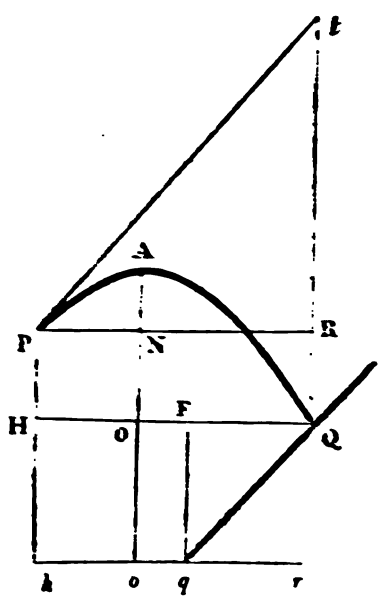
If $z = 45°$, then $\sin. 2z = \text{rad.} = 1$; $\cos. z = \sin. z = \frac{1}{\sqrt{2}}$, and $\therefore \overline{\cos. z}^2 = \overline{\sin. z}^2 = \frac{1}{2}$; $\tan. z = \text{rad.} = 1$: hence these expressions become

$$R = p + \sqrt{p^2 - 2ph}$$

$$A = \tfrac{1}{2}p - h$$

$$T = \sqrt{\frac{R + h}{m}}.^{a}$$

5. If, instead of descending upon the *horizon* (*hor*), the body falls upon a *plane* (Qq) inclined to it in some *given angle* Qqr (β), then draw qF at right $\angle^{r}$ to HQ. and let $hq = HF = r$, $Ph = h'$, and $Qq = x$, then will $FQ = x \times \cos. \beta$, and $qF = Hh = x \times \sin. \beta$; hence $HQ = HF + FQ = r + x \times \cos. \beta$, and PH



(a) Hence $T^2 = \frac{R+h}{m}$, and $R + h = mT^2$ = space described in T''.

$PH\,(h) = Ph - Hh = h' - x \times \sin.\beta$; substitute these values for HQ and h in the equations marked (B) and (C), and we have,

$$r + x \times \cos.\beta = p \times \sin.2\alpha + \sqrt{4p \times \overline{\cos.\alpha}^2 . p \times \overline{\sin.\alpha}^2 + h' - x \times \sin.\beta} \quad (D)$$

$$Ao = p \times \overline{\sin.\alpha}^2 + h',$$

$$T = \sqrt{\frac{r + x \times \cos.\beta . \tan.\alpha + h' - x \times \sin.\beta}{m}},$$

in which the value of x being found from the *first* equation, the value of T will also be known.

6. The equation marked (D) may be rendered more simple from the particular circumstances under which the body is projected; for instance,

I. Let $\alpha = 0$, i. e. let the body be projected in a line *parallel to the horizon*, then $\sin.\alpha = 0$, $\sin.2\alpha = 0$, and $\cos.\alpha = 1$, $\therefore$ the equation becomes

$$r + x \times \cos.\beta = \sqrt{4p . h' - x \times \sin.\beta}.$$

II. Let $\alpha = 45^\circ$, then $\sin.2\alpha = 1$, $\overline{\sin.\alpha}^2 = \frac{1}{2}$, $\overline{\cos.\alpha}^2 = \frac{1}{2}$, $\therefore$ the equation is,

$$r + x \times \cos.\beta = p + \sqrt{2p . \tfrac{1}{2}p + h' - x \times \sin.\beta}.$$

III. Let

T'' (which call S), then $h = S - R$; if the *range* and *time of flight* therefore be given when $\alpha = 45^\circ$, the height of the point P may be found *by subtracting R from the space described in that time.* If $R = h$, then $2h = S$, and $h = \frac{1}{2}S = \frac{1}{2}$ space described in T''; i. e. if a body be projected in an $\angle$ of 45° from the *top* of a building whose height is (h), and after T'' falls at a distance from the *bottom* equal to that height, then the height of the building is equal to *half the space described in T''.*

III. Let $r=0$, then

$$x\times\cos.\beta=p\times\sin.2\alpha+\sqrt{4p\times\overline{\cos.\alpha}^{2}\cdot p\times\overline{\sin.\alpha}^{2}+h'-x\times\sin.\beta},$$

is the equation to determine the value of x when the plane rises immediately from the extremity of the vertical line Ph.

IV. Let $h'=0$, then

$$r+x\times\cos.\beta=p\times\sin.2\alpha+\sqrt{4p\times\overline{\cos.\alpha}^{2}\cdot p\times\sin.\alpha^{2}-x\times\sin.\beta},$$

is the equation to determine the value of x, when the body is projected from a *point in the horizon*, and falls upon a plane ascending from the horizon at some given distance (hq) from the point of projection.

By way of conclusion, it may be observed, that the equation marked (D) will also apply to the case of a body projected in a direction *below* the line PR, merely *by changing the sign of* 2α; for when Pt is *below* PR, sin. α (and consequently sin. 2α) becomes *negative*, but the signs of $\overline{\cos.\alpha}^{2}$ and $\overline{\sin.\alpha}^{2}$ remain the *same*, whether cos. α and sin. α be *positive* or *negative*.

EXAMPLE.

At the foot of a tower 60 *feet high, runs a river whose breadth is* 300 *feet; a hill slopes from the opposite side of the river in an angle of* 45° *to the horizon; from the top of the tower a cannon ball is fired in a direction making an angle of* 60° *with a line drawn parallel to the horizon; the velocity of projection*

projection is such as a body would acquire in falling 500 *feet; At what distance from the bank of the river will the ball strike the hill?*

This is a case of Art. 5. page 220, where $r=300$; $h'=60$; $p=500$; $a=60^\circ$; $\therefore$ sin. $a=\frac{\sqrt{3}}{2}$, sin. $2a=$ sin. $120^\circ=\frac{\sqrt{3}}{2}$, and cos. $a=\frac{1}{2}$; $\beta=45^\circ$, $\therefore$ sin. $\beta=\frac{1}{\sqrt{2}}$, and cos. $\beta=\frac{1}{\sqrt{2}}$; substituting these values for the different quantities in the general equation, we have

$$300+\frac{x}{\sqrt{2}}=500\times\frac{\sqrt{3}}{2}+\sqrt{2000\times\tfrac{1}{4}\times500\times\tfrac{1}{4}+60-\frac{x}{\sqrt{2}}},$$

which *reduced*, is

$x^2+331x=399656$, very nearly;

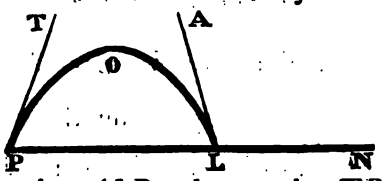
$\therefore x=487\frac{1}{2}$, very nearly.

XIX.

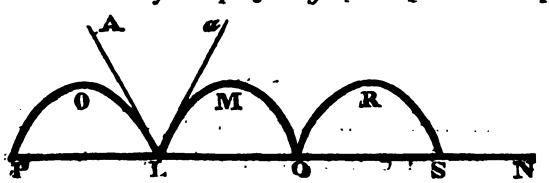
In determining the motion of projectiles in this Lecture, we have not taken into consideration the nature of the bodies thus projected (as to their *hardness* or *elasticity*), nor of that of the planes upon which they impinged. In making a few observations upon this subject, we shall follow the order observed in Sect. XXIV. page 188.

1. If a *perfectly hard* body be projected along a *perfectly hard* horizontal plane (PLN), and AL be a *tangent*

a *tangent* to the parabola (POQ) at the point L, it will impinge upon the plane in the direction AL, under precisely the same circumstances as the body in Art. 1. page 188; and since the velocity *at the point* L is the same as the velocity *at the point* P[a], and the angle ALP = the angle TPL, it will afterwards proceed along the plane with a *velocity* which is to the velocity of *projection* :: cos. $\angle ALP$ or TPL : radius :: cosine of the *angle of projection* : radius.



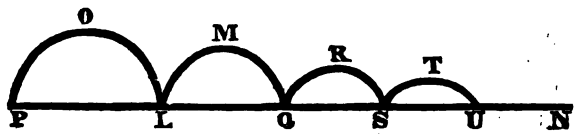
2. If the body be *perfectly elastic*, and the plane perfectly hard, then, by Art. 2. page 189, it will *ascend* from the plane with the *same velocity*, and under



(a) Since the velocity in the parabola at any point is equal to velocity acquired in falling down the *parameter* to that point, it *varies* as $\sqrt{\text{parameter}}$, and consequently is the same *at all points equally distant from the focus.*

under the *same angle* as that in which it impinges upon the plane; and as this *velocity* and this *angle* is the same with the velocity and angle of *projection*, it will describe the *same parabola*; and it will go on for ever describing the equal parabolas *POL*, *LMQ*, *QRS*, &c. till acted upon by some extraneous force.

3. If the body or the plane, or *both*, be *imperfectly elastic*, and *perfect* : *imperfect* elasticity :: 1 : m, then, by Art. 3, page 190, it will ascend from the plane in an angle *lesser* than *the angle of projection* (α), and with a velocity which is to the *velocity of projection* :: $\sqrt{\cos.\alpha^2 + m^2 \times \sin.\alpha^2}$: 1. Now the *horizontal range* of a projectile varies as $v^2 \times$ *the sine of twice the angle of projection* (Art. 3. page 205);



in this case therefore it would go on to describe a series of parabolas *POL*, *LMQ*, *QRS*, &c. whose *ranges* *PL*, *LQ*, *QS*, &c. *decrease* in a pretty high geometrical ratio, and consequently their *sum* $PL + LQ + QS +$ &c. soon comes to its limit. Hence, whatever may be the velocity or angle of projection, the *whole range* described by the body will be *finite*, and its motion cease after a definite time.

The

The actual motion of a projectile near the earth's surface must in course be considered under our *third* case, as there exist not in nature bodies which are *perfectly hard* or *perfectly elastic*. If the angle of elevation and velocity of projection be considerable, the *direction* in which it returns to the earth, and the velocity with which it moves, will be such as to cause the body to penetrate into the earth's surface, and its motion will be destroyed. But if the angle of elevation be *small*, and the velocity of projection considerable, it will then strike the earth in such a manner as to destroy only a *part* of its velocity, and it will *rebound* so as to describe a series of *decreasing parabolas* similar to those in Art. 3, the number and dimensions of which will depend upon the circumstances under which it is projected and the nature of the soil upon which it impinges. But to trace the motion of a projectile of this description, or to estimate the deviation of its path from the parabolic shape on account of the resistance which it meets with from the air, &c. &c. belongs not to a treatise of this kind, but to those books which treat professedly of "*the Art and Practice of Gunnery.*"

AN

INTRODUCTION

TO THE

STUDY OF NATURAL PHILOSOPHY.

PART II.

ON MECHANICAL ACTION AND PRESSURE;
AND THE POWERS OF MACHINERY.

Lecture VII.

ON THE LEVER.

In the preceding Lectures, the motion of bodies has been supposed to arise either from collision, or from the *immediate* action of one or more forces. We now proceed to consider the effects produced, when these forces are made to act by the *intervention* of other bodies. These intermediate bodies are called *Machines;* and by means of them the effect of a given force may be increased or diminished in any given ratio. Machines are divided into *simple* and *compound.* The *simple* machines, or (what are more commonly called) the *mechanical powers,* are six in number; viz. 1. The *Lever;* 2. The *Wheel and Axle;* 3. The *Pulley;* 4. The

4. The *Inclined Plane;* 5. The *Screw*; 6. The *Wedge*[a]. *Compound* machines are formed from the combination of two or more *simple* ones. But it is not the object of this treatise to enter upon a full description of the nature and use of compound machinery; our intention is rather to explain, upon mathematical principles, the general theory of mechanical action. We shall therefore begin with estimating the relation between the forces acting upon the arms of a *straight lever*, which, of all the mechanical powers, is the *most simple*.

XXX.

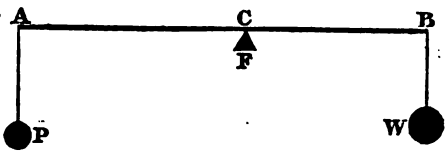
On the relation of the forces which keep each other in equilibrio upon a straight lever.

DEFINITION.—The *Lever* is an inflexible bar or rod, some point of which being supported, the rod itself is moveable freely round that point, as a *center of motion*.

1. Let *ACB* be a *straight lever*, supported by a prop or *fulcrum F*, and moveable about the point *C* as its center

(a) In philosophical strictness, the number of simple machines may be reduced to *three;* viz. the *lever*, the *inclined plane*, and the *cords* or *ropes* which connect the power and weight with the different parts of the machine; for the mechanism of the *wheel* and *axle*, and of the *pulley*, merely combines the principle of the *lever* with the tension of *cords;* the properties of the *screw* depend entirely on those of the *lever* and the *inclined plane;* and the case of the *wedge*, so far as it is capable of mathematical demonstration, is very analogous to that of a body sustained between two *inclined planes*.

center of motion. From the extremities of its arms *CA, CB*, let two weights *P, W*, be suspended; and suppose them to be in equilibrio about *C*, the lever itself remaining in an horizontal position. In the present instance,



let us also suppose that the lever *AB*, and the cords *AP, BW*, by which the weights are suspended, are entirely void of gravity; in which case it is evident that the equilibrium of the bodies does not at all depend upon the *length* of the cords *AP, BW;* and as (by Art. 8. page 101) the centers of gravity of the bodies *P, W* are in the direction of the lines *AP, BW*, the effect will be the same whether the bodies be *suspended* by the strings *AP, BW*, or whether they be placed *with their centers of gravity in the points A, B respectively*[b]. In this latter case the point *C* becomes the *common center of gravity* of the weights *P, W*, and consequently (by Art. 5. page 97.) *P* : *W* :: *BC* : *AC*; i. e. "when two weights are in equilibrio "upon a *straight lever*, "placed in an horizontal position, they are to each "other *inversely* as the length of the arms from "which they are respectively suspended."

2. But

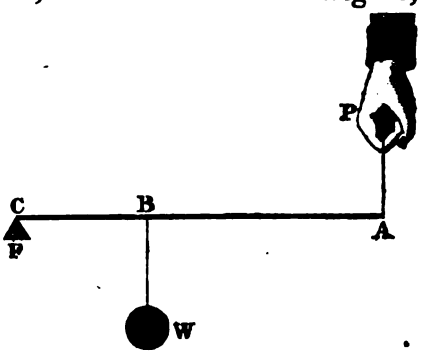
(b) For since the equilibrium does not at all depend upon the length of the lines *AP, BW*, we may suppose those lines to *vanish*; in which case the centers of gravity of *P, W* may be considered as coinciding with the extremities *A, B* of the lever.

2. But it is evidently quite immaterial to the truth of the foregoing demonstration, whether the equilibrium of the lever be produced by the *force of gravity* of the two weights P, W, or by the action of *any other forces* in the directions AP, BW; we may therefore assume it as a general principle, that "if two forces, acting in the *same plane*, and *perpen-* "*dicular* to the extremities of a straight lever, be in "*equilibrio*, they will be to each other *inversely* as the "arms upon which they respectively act;" and the effect of these forces to turn the lever about the center of motion will be measured by the *product arising from multiplying each force into the distance at which it acts from that center*; (See Note *a*, page 98;) for if the magnitude of the forces acting at A and B be represented by P and W respectively, then (since $P : W :: BC : AC$) $P \times AC = W \times BC$; $\therefore P \times AC$ represents the effect of P, and $W \times BC$ represents the effect of W, to turn the lever round C[a].

3. Levers

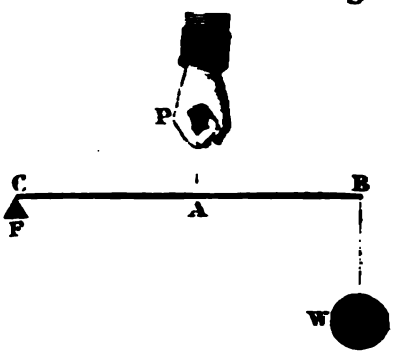
(a) From hence it appears, that "*any number of weights will* "*keep each other in equilibrio upon the arms of a straight lever,* "*when the sums of the products arising from multiplying each* "*weight by its distance from the center of motion, are equal on* "*each side that center*;" for let AD (Fig. in page 108) represent a straight lever, whose center of motion is G, and let the bodies or weights A, B, C, D be placed upon its arms AG, DG, at different distances from G; then the effort of A to turn the lever about G being represented by $A \times AG$, of B by $B \times BG$, of C by $C \times CG$, &c. the whole effect upon the *arm* AG will be represented by $A \times AG + B \times BG$, and upon the *arm* DG by $C \times CG + D \times DG$; there will consequently be an equilibrium when $A \times AG + B \times BG = C \times CG + D \times DG$.

3. Levers are divided into three different *orders*, according to the position of the *power* and *weight* with respect to the *center of motion*. I. In a lever of the *first* order, *the center of motion is between the power and weight*, as in the preceding instance, where P and W may bear all possible relations to each other, and where the *pressure upon the fulcrum* (since P and W press in the *same* direction) will be proportional to $P + W$. II. In a lever of the *second* order, *the weight is placed between the power and the center of motion*, as in the annexed Figure, where the weight W is supported by the power P acting *upwards* in the direction AP. In this case, when there is an equilibrium, $P : W :: BC : AC$ [b], P is therefore *less* than W; and the *pressure upon the fulcrum* (P and W pressing in *opposite* directions) is



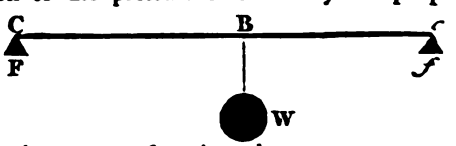
(b) That this proportion between P and W holds also when they are on the *same* side of the fulcrum, may be thus proved; the effort of the weight W to turn the lever about C is measured by $W \times BC$; the effort of the power P (acting in direction PA) to turn the lever about C, i. e. *to sustain* W, is measured by $P \times AC$; when there is an equilibrium therefore, $P \times AC$ must be equal to $W \times BC$, or $P : W :: BC : AC$, as before.

is proportional to $W-P$ [a]. III. In a lever of the *third* order, *the power P acts between the weight W and the center of motion C*; and since *BC* is *greater* than *AC*, *P is greater than W*, and the *pressure upwards from the fulcrum* is represented by $P-W$.[b]



4. The nature of many of the common machines and instruments made use of for moving and sustaining

(a) From the principle of a lever of the *second* order, we may estimate the proportion of the pressure sustained by two props *F*, *f*, when the weight *W* is supported by a rod *Cc* placed upon them. For suppose



C, *c* to be successively the centers of motion, then

Pressure on fulcrum *f* : weight *W* :: *BC* : *Cc*; for same reason,

Weight *W* : *Pressure* on fulcrum *F* :: *Cc* : *Bc*; ∴ ex æquo,

Pres. on fulcrum *f* : *Pressure* on fulcrum *F* :: *BC* : *Bc*; and as the *whole weight* is sustained by the two props, it is divided between them in the ratio of *BC* : *Bc*, the part sustained by each prop being *inversely* as the distance of the weight from it.

(b) In this *third* order of levers, although the lever is supposed to move *freely* round the center of motion *C*, it is yet necessary to consider it as *firmly connected* with the prop at that point.

taining weights, and for overcoming resistances, may be explained upon the principle of the lever. Thus, not only the *common balance*, the *steelyard*, &c. but all such instruments as *scissars*, *snuffers*, *pincers*, &c. (where a power is applied at one end of the instrument for the purpose of overcoming the resistance opposed to it at the other) may be reckoned amongst *levers of the first order*. The *instruments used for moving heavy weights*, whereof one end rests upon the ground whilst a power is applied at the other; the *oars of boats*, which may be considered as changing their *center of motion* in the water after every impulse which they receive from the rower; *cutting-knifes*, fixed at one end, and made to act by means of the pressure of the hand upon the other; *doors*, whereof the centers (or rather the *axes*) of motion are in the hinges; &c. &c., all come under the denomination of levers of the *second*[c] order. A *ladder*, raised up by means of a power applied towards its lower end, in which case the principal part of its weight is thrown *beyond* the point where the power acts; *sheers* (used for clipping sheep), in which the pressure of the hand is applied between the center of motion and the part where the resistance operates; the *human arm*, raising by its muscular action a weight suspended at one extremity of it, the center of motion being at the other; &c. &c., are instances of levers of the *third* order.

5. Let

(c) For in all these instances it is evident that the weight to be moved, or the resistance to be overcome, is *between the center of motion and the point where the power acts*.

5. Let us next estimate the relation of the forces which keep a lever in equilibrio, when its *own weight* is taken into consideration. In this case it is evident, that in a lever of the *first* order the effect of the *power and weight* (and in levers of the *second* and *third* orders the effect of the *weight only*) will be increased by the *weight of the arm* upon which they respectively act; and since this weight may be considered as collected in the *center of gravity* of the arm, (Art. 4, page 95,) this additional force will be measured by *the weight of the arm multiplied into the distance of its center of gravity from the fulcrum.* Suppose the lever to be of a *cylindric* or *prismatic* form, and that its weight $= w$, then,

In a lever of the first order (Fig. page 229), the weight of the arm $AC = \frac{w \times AC}{AB}$, weight of the arm $BC = \frac{w \times BC}{AB}$[a]; and as the centers of gravity of AC and BC are in their *middle points*, the distance of the center of gravity of AC from the fulcrum $= \frac{1}{2}AC$, and that of $BC = \frac{1}{2}BC$; hence the effect of the power P is increased by a force which is measured by $\frac{w \times AC}{AB} \times \frac{1}{2}AC \left(\text{or } \frac{\frac{1}{2}w \times AC^2}{AB} \right)$, and the effect of W by a force measured by $\frac{w \times BC}{AB} \times \frac{1}{2}BC$,

(a) For $AB : AC ::$ whole weight of the lever (w) : weight of the arm $AC = \frac{w \times AC}{AB}$; and $AB : BC ::$ whole weight of lever (w) : weight of the arm $BC = \frac{w \times BC}{AB}$.

$\frac{1}{2}BC$, $\left(\text{or } \frac{\frac{1}{2}w \times BC^2}{AB}\right)$. The effects of P and W are measured by $P \times AC$ and $W \times BC$ respectively; in the case of an equilibrium, therefore, we have,

$$P \times AC + \frac{\frac{1}{2}w \times AC^2}{AB} = W \times BC + \frac{\frac{1}{2}w \times BC^2}{AB},$$

$$\text{or } P = \frac{W \times BC}{AC} + \frac{\frac{1}{2}w \,.\, \overline{BC^2 - AC^2}}{AB \times AC}.$$

In a lever of the second order (Fig. page 231), the *whole weight* of the lever operates in direction of W; $\therefore$

$$P \times AC = W \times BC + w \times \tfrac{1}{2}AC,$$

$$\text{or } P = \frac{W \times BC}{AC} + \tfrac{1}{2}w.$$

In a lever of the third order (Fig. page 232), the whole weight of the lever also operates in conjunction with W; $\therefore$

$$P \times AC = W \times BC + w \times \tfrac{1}{2}BC,$$

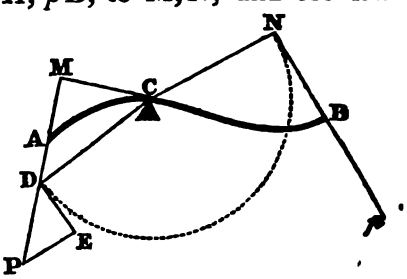
$$\text{or } P = \frac{\overline{W + \frac{1}{2}w} \times BC}{AC}.$$

XXXI.

On the general Properties of the Lever, and a Combination of Levers.

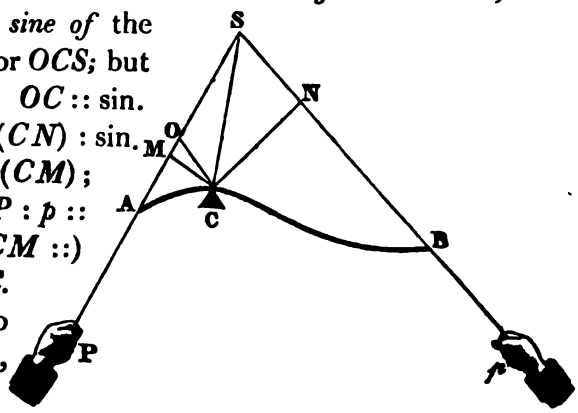
1. Let ACB be any lever, whose center of motion is C; and let two forces P, p act in the direction P, p upon the extremities of its arms CA,

CA, CB. Produce *PA, pB,* to *M, N,* and let fall the perpendiculars *CM, CN;* with the *longer* perpendicular *CN* and center *C* describe the circular arc *ND,* and join *CD.* Let *DP* represent the magnitude of the force acting upon the lever at *A* in direction *AP,* and let it be resolved into two others, viz. *DE perpendicular* and *EP parallel* to the radius *CD;* then *DE* only is effectual to produce motion round the center *C,* the part *EP* being exerted merely to produce pressure upon the fulcrum in direction *CD.* Supposing, therefore, the lever to be perfectly inflexible, this effort of *P* to produce motion round *C* would be counteracted by a force equal to *DE,* applied perpendicularly at *N,* in direction *NB.* Let then the power *p* be equal to that part of *P* which is represented by *DE,* and (as it is indifferent in what point of the line of direction this power acts) conceive it to act at *N.* In this case the forces *P, p* will be in equilibrio; i. e. when $P : p :: PD : DE$, the lever will be kept in equilibrio about the center of motion *C;* but by similar triangles $PD : DE :: CD\ (CN) : CM$, $\therefore P : p :: CN : CM$; hence "two forces acting "at the extremities of the arms of any lever will "be in equilibrio, when they are to each other "*inversely* as the perpendiculars let fall upon the "lines of direction in which they respectively act."



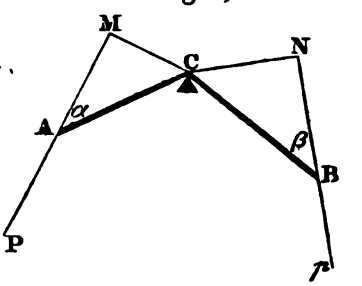
2. Produce

2. Produce PA, pB till they meet in S; join CS, and draw CO parallel to pS, in which case the angle $OCS = CSN$. Now if CS be made *radius*, CM becomes the *sine of the* $\angle CSO$, and CN the *sine of* the $\angle CSN$ or OCS; but as $SO : OC ::$ sin. $\angle OCS\,(CN)$: sin. $\angle CSO\,(CM)$; hence $P : p ::$ $(CN : CM ::)$ $SO : OC$. The two sides SO, OC of



the triangle SOC represent, therefore, the relative magnitude and direction of the two forces P, p; the *third side* SC will consequently represent a force equivalent to them both (Art. 6. page 50); and as this compound force acts directly towards C, the *pressure* upon the fulcrum will be represented, in quantity and direction, by the line SC.

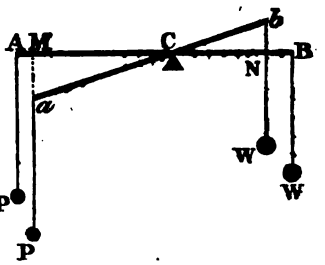
3. If each arm of the lever be *straight*, but the two arms inclined to each other in the *given angle* ACB, let $\angle CAM = \alpha$, $\angle CBN = \beta$, and rad. $= 1$; then $AC : CM ::$ rad. (1) : sin. α, $\therefore CM = AC \times$ sin. α, and $BC : CN$



:: rad. (1) : sin. β, $\therefore CN = BC \times$ sin. β; hence $P : p$

$P : p :: BC \times \sin.\ \beta : AC \times \sin.\ \alpha$. If $\sin.\ \alpha = \sin.\ \beta$, then $P : p :: BC : AC$, or "*P, p are to each other as the arms of the lever upon which they respectively act.*"

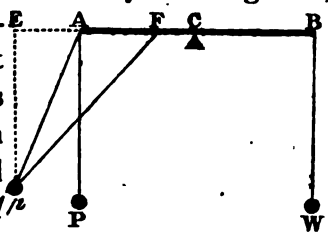
4. If the lever be straight, and the forces act parallel to each other, then $\sin.\ \alpha = \sin.\ \beta$, $\therefore P : W :: BC : AC$, as in Art. 2, page 230; and this will be the case whatever be the position of the lever; if therefore P and W be in equilibrio when the lever is in the horizontal position ACB, they will also be in equilibrio when it is in any other position aCB; i. e. the lever thus acted upon *will rest in any position*(a).



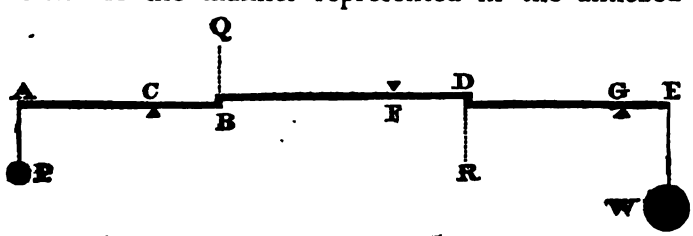
5. In the *common balance* or *scales*, the arms AC, CB are equal to each other; $\therefore$ when there is an equilibrium, $P = W$. But this equilibrium will be destroyed, if either P or W be removed from their parallel

(a) This also appears from Art. 1. page 235; for produce Pa to M, then, when the lever is in position aCb, $P : W :: CN : CM$:: (by sim. Δ^s) $Cb : Ca$:: (since the arms of the lever are invariable) $CB : CA$. In thus asserting that the lever will *rest in any position*, it is in course taken for granted that the *common center of gravity of P, W and the lever* coincides with the *center of motion*; for it is evident, from the principles laid down in Lect. 4. Part I. that the lever will only rest when that center is *supported*.

parallel position. Suppose, for instance, a person placed in the scale P was balanced by the weight W, but, by pushing in the oblique direction pF against the arm CA, the scale was protruded into the position pA; then draw pE parallel to PA, and produce CA^n to meet it in E; and at the instant of the scales arriving at the position pA, it will act at the *perpendicular* distance CE from the center of motion; its effect therefore (by Art. 2) to turn the lever about C will be measured by $P \times CE$, i. e. its effect will exceed that of W in the proportion of $CE : CA$, and consequently the scale in which the person is will *preponderate*, and the equilibrium be destroyed.



6. In a combination of levers connected with each other in the manner represented in the annexed



Figure, there will be an equilibrium when $P : W :: BC \times DF \times EG : AC \times BF \times DG$. For suppose the equilibrium to exist, and that the forces which act at B, D are represented by Q, R respectively, then

 $P : Q$

$$P : Q :: BC : AC \text{ or } Q = \frac{P \times AC}{BC},$$
$$Q : R :: DF : BF$$
$$R : W :: EG : DG \text{ or } R = \frac{W \times EG}{DG};$$

$\therefore$ ex æquo $P : W :: BC \times DF \times EG : AC \times BF \times DG$ (M).

Also,

the *pressure* on $C = P + Q = P + \frac{P \times AC}{BC} = \frac{P \times AB}{BC}$,

- - - - on $F = Q + R = \frac{P \times AC}{BC} + \frac{W \times EG}{DG}$,

- - - - on $G = R + W = \frac{W \times EG}{DG} + W = \frac{W \times DE}{DG}$.

We have here supposed the forces to act perpendicular to the extremities of the several levers; if they acted *obliquely*, or if the arms of the levers were *inclined to each other*, then for these arms must be substituted, in the proportion marked (M), the perpendiculars let fall from the centers of motion, E, F, G, upon the direction in which the forces act. We now proceed to illustrate the foregoing theory by a few plain examples.

EXAM. I.

At the extremities of a straight lever, whose length is 24 *inches, are placed two weights of* 5 *and* 7 *pounds; At what point must the fulcrum be placed so that these weights shall balance each other, the weight of the lever not being taken into the account?*

This is a case of the lever of the *first* order, in which (Fig. page 229) $P = 5$, $W = 7$, and $AB = 24$; and

and when there is an equilibrium, P (5) : W (7) :: BC : AC; $\therefore$ 12 : 7 :: $BC+AC$ ($AB=24$) : $AC=\frac{168}{12}=14$; hence $BC=24-14=10$. inches.

EXAM. II.

At the extremity of a lever of the second order, there acts a power which is of itself able to sustain only a weight of 15 *pounds; but when acting under this mechanical advantage, it is able to sustain a weight of* 100 *pounds, placed* 5 *feet from it; what is the length of the lever?*

Referring to Fig. page 231, we have $P=15$, $W=100$, and $AB=5$ feet; now P (15) : W (100) :: BC : AC, $\therefore$ P : $W-P$ (or 15 : 85) :: BC : $AC-BC$ ($AB=5$); hence $BC=\frac{5\times 15}{85}=\frac{15}{17}$, and $AC=AB+BC=5\frac{15}{17}$ feet.

EXAM. III.

A body, suspended at the extremities of a balance whose arms are unequal, weighs (p) *pounds at one end, and* (q) *pounds at the other; what is its real weight?*

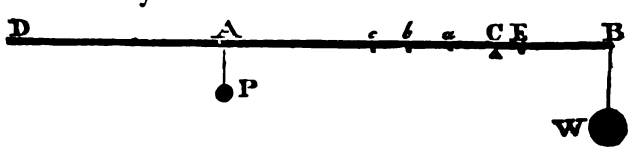
A balance of this kind is called a *false* balance, because when the body is suspended at the extremity of the longer arm, the weight which balances it is *above*, and when suspended at the extremity of the *shorter* arm, is *below*, the *true* weight. But the true weight of the body is easily found by the following operation; viz. Let $x=$ the true weight, and a the arm of the lever upon which (p) is suspended to balance

balance it, and b the arm upon which (q) is suspended to balance it; then $x : p :: a : b$, or $x = \frac{pa}{b}$, and $x : q :: b : a$ or $x = \frac{qb}{a}$; multiply these two equations together, and we have $x^2 = pq$ or $x = \sqrt{pq}$; i. e. "the "*true* weight of the body is a *mean proportional* "between the *apparent* weights thus obtained."

EXAM. IV.

To explain the construction of the steelyard.

A *steelyard* is a lever of the first order, having two *unequal* arms BC, CD; a *given* weight P is moveable along the *longer* arm CD, so as to sustain a weight of a *variable* magnitude suspended from the extremity of the *shorter* arm CB.



1. Suppose the *weight of the arms* not to be taken into consideration, then $P : W :: BC : AC$ (Art. 1. page 229), $\therefore W \times BC = P \times AC$, and as P and BC are *given*, $W \propto AC$. Hence, if Ca, ab, bc, &c. be taken *equal* to each other (or $Cb = 2Ca$, $Cc = 3Ca$, &c.), then if P balances *one* pound when placed at a, it will balance *two* pounds at b, *three* pounds at c, &c. &c.

2. Let us next suppose the steelyard to have *weight*, and that the excess of the weight of the *longer* arm CD above that of the *shorter* CB is such that

that the moveable weight P, when placed at E, would keep the arms in equilibrio; in which case this *excess* would be measured by $P \times CE$; if therefore a weight W placed at B be in equilibrio with the weight P placed at A, we should have $W \times BC = P \times AC + P \times CE = P.\overline{AC + CE} = P \times AE$; consequently, since P and BC are *given*, $W \propto AE$. The construction of the steelyard, therefore, would be the same as in the former instance, except that the graduation must begin from E, instead of from C.

3. *In general*, let the weight of the arm $CD = w$, and the distance of its *center of gravity* from $C = g$; the weight of the arm $CB = w'$, and the distance of its *center of gravity* from $C = g'$; let $CB = m$, and Ca, Cb, Cc, &c. $= d$, d', d'', &c. respectively; and let W, W', W'', &c. represent the weights sustained by P when placed at the points a, b, c, &c. successively; then (by Art. 5, page 235,) the following equations will be true; viz.

$$Wm + w'g' = Pd + wg \ (A)$$
$$W'm + w'g' = Pd' + wg \ (B)$$
$$W''m + w'g' = Pd'' + wg \ (C)$$
$$\&c. = \&c.$$

Subtract (A) from (B), (B) from (C), &c. &c. then

$$\overline{W' - W}.m = P.\overline{d' - d}, \text{ or } d' - d \ (ab) = \frac{\overline{W' - W}.m}{P}$$

$$\overline{W'' - W'}.m = P.\overline{d'' - d'}, \text{ or } d'' - d' \ (bc) = \frac{\overline{W'' - W'}.m}{P}$$

&c. = &c. &c. = &c.

If

If W, W', W'', &c. be in *arithmetical progression*, then $W' - W = W'' - W' =$ &c. $\therefore ab = bc =$ &c.; and if $W' - W$, $W'' - W'$, &c. $= n$, then ab, bc, &c. are each equal to $\frac{nm}{P}$. Hence this general rule for making the divisions upon the graduated arm of a steelyard; viz. "Let m = the length of the *shorter* "arm, and let some given weight (P) balance "another given weight (W) at the given distance "(Ca) upon the *longer* arm; let n = the *least* "*difference* of weights which the steelyard is "required to ascertain, and make the divisions, "ab, bc, &c. each equal to $\frac{nm}{P}$; the weight (P) "suspended from the points a, b, c, &c. will balance "W, $W+n$, $W+2n$, &c."(a) Thus suppose $P = 1lb.$ $W = 2lbs.$ and $n = \frac{1}{4}lb.$, then $\frac{nm}{P} = \frac{1}{4}m$; if therefore each division ab, bc, &c. is taken equal to $\frac{1}{4}m$, the steelyard will serve for ascertaining weights *above* $2lbs.$ proceeding by *quarters of a lb.*; if $n = 1$ ounce ($= \frac{1}{16}lb.$), then, by taking ab, bc, &c. each equal to $\frac{1}{16}m$, it will serve for ascertaining the *ounces*; &c. &c.

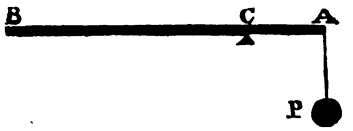
EXAM. V.

ACB is a cylindrical straight lever whose weight is (w), at the extremity A a given weight (P) is suspended;

(a) If W be greater than W', then n is *negative*; and by carrying on the graduation backwards from a towards C, P will be made to sustain $W-n$, $W-2n$, $W-3n$, &c.

pended; it is required to determine the position of the fulcrum C, so that P may be in equilibrio with the longer arm BC.

Let $AB = a$,
$AC = x$,
then $BC = a - x$,
W^t. of $AC = \frac{wx}{a}$,
W^t. of $BC = \frac{w.\overline{a-x}}{a}$



Hence, from the principles laid down in Art. 5, page 235, when there is an equilibrium, we have,

$$P \times x + \frac{wx}{a} \times \tfrac{1}{2}x = \frac{w.\overline{a-x}}{a} \times \tfrac{1}{2}.\overline{a-x},$$

$$\text{or } Px + \frac{wx^2}{2a} = \frac{w.\overline{a-x^2}}{2a},$$

$$\therefore 2aPx + wx^2 = wa^2 - 2awx + wx^2,$$

$$\text{and } 2.\overline{P+w}.x = wa,$$

$$\text{or } x = \frac{wa}{2.\overline{P+w}}.$$

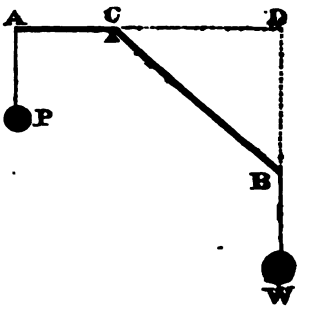
Cor. If $P = w$, then $x = \frac{1}{4}a$, or $AC = \frac{1}{4}AB$ the length of the lever.

EXAM. VI.

P and W are suspended from the extremities of the arms of the bent lever ACB (whose weight is not taken into the consideration); it is required to find the angle of inclination (ACB), so that when there is an equilibrium, AC shall be parallel to the horizon.

By

By Art. 3, page 237, $P : W :: BC \times \text{sin. } CBD : AC \times \text{sin. } \angle PAC$; but sin. $\angle PAC = \text{rad.} = 1$, and sin. CBD = cos. $\angle BCD$; $\therefore P : W :: BC \times \text{cos. } BCD : AC \times 1$, hence cos. $BCD = \frac{P \times AC}{W \times BC}$, from which $\angle BCD$ (and consequently ACB, which is equal to $180^\circ - BCD$) is known.



Cor. 1. If $P = W$, then cos. $BCD = \frac{AC}{BC}$; and $BC = 2AC$, then $\frac{AC}{BC} = \frac{1}{2}$; $\therefore$ cos. $BCD = \frac{1}{2} =$ cos. 60°; hence $ACB = 120^\circ$.

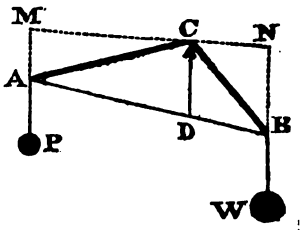
Cor. 2. If $AC = BC$, then cos. $BCD = \frac{P}{W}$; and if $W = 2P$, then $\frac{P}{W} = \frac{1}{2}$; $\therefore$ cos. $BCD = \frac{1}{2} =$ cos. 60°; hence, in *this* case also, the $\angle ACB$ is 120°.

EXAM. VII.

From the extremities of the arms CA, CB of a bent lever, the weights P, W are suspended; it is required to determine the position of the lever when these weights are in equilibrio.

Through C draw MN parallel to the horizon, and produce PA, WB to meet it in M and N; by Art. 1,

Art. 1, page 236, the lever will be in equilibrio, when $P : W :: CN : CM$. Join AB, and draw CD parallel to MP or NW; CD will cut AB in the same proportion that it does MN (Euc. 6. 10), i. e. $DB : AD :: CN : CM$; hence $DB : AD :: P : W$; from which it appears that "if the line AB, which "joins the extremities of "the arms of the lever, be "divided in the point D, in "the ratio of $P : W$, and "that point be brought "immediately under the "center of motion, it will give the position of the "lever when P and W are in equilibrio."

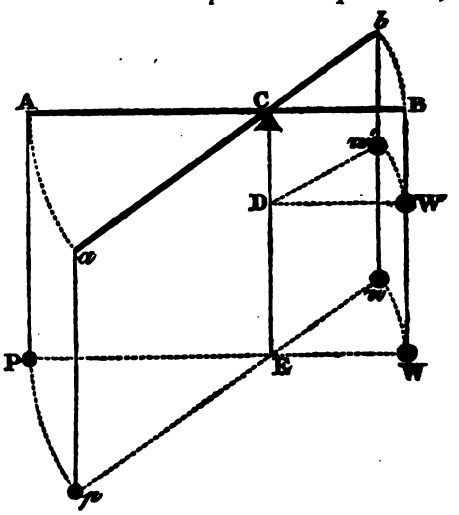


EXAM. VIII.

If from the arms of a straight lever, ACB, two weights P, W be suspended, and the lever be made to turn round its center of motion (C), then will these two weights describe similar circular arcs, whether they be suspended by equal *strings AP, BW, or by* unequal *strings AP, BW'.*

1. Let us first suppose them to be suspended by *equal* strings AP, BW, and let the lever be turned from its *horizontal* position ACB to *any other* position aCb; the weight P in the mean time describing the line Pp, and the weight W to the line Ww. Complete the parallelograms $ACEP$, $CEWB$, (which will evidently form *one* parallelogram $APWB$) and join Ep, Ew; then (since $ap = AP = CE$)

$= CE$) ap and CE will be *equal and parallel*, $\therefore$ (Euc. 1. 33) $CapE$ is a *parallelogram*. In the same manner it may be proved that $CEwb$ is a parallelogram, and that $CapE$, $CEwb$ form *one* parallelogram $apwb$. Now $Ep = Ca = CA = EP$, $\therefore$ Pp is a *circular arc* whose center is E, and radius EP or CA; for same reason Ww is a circular arc whose center is E and radius EW or CB; and since the angle WEw is equal to the angle PEp, the arcs Pp, Ww are *concentric* and *similar*.



2. Suppose now the string BW' *not to be equal* to the string AP; then complete the parallelogram $CBW'D$, and (w' being the position of W' when the lever is in position aCb) join Dw'. In the same manner as before, it may be proved that W' describes a circular arc $W'w'$, whose center is D, and radius DW' or CB; but it is evident that DW' is equal to EW (being opposite sides of a parallelogram), the arc $W'w'$ is therefore equal and similar to the arc Ww; hence "P and W describe similar circular arcs "whilst

" whilst the lever moves from the position *ACB* " to the position *a Cb*, whether the strings by which " they are suspended be *equal* or *unequal*."

XXXII.

QUESTIONS FOR PRACTICE.

1. At one extremity of a straight lever whose length is 7 feet, a weight of 10 pounds is suspended; at the distance of 2 feet from the point of suspension a fulcrum is placed; What weight must be suspended from the other extremity of the lever, to keep it in equilibrio? Ans. 4*lbs.*

2. A lever of the second order is 20 feet long; at what distance from the fulcrum must a weight of 112*lbs.* be placed, so that it may be supported by a power (able to sustain) 50 pounds, acting at the extremity of the lever?

Ans. 8 feet and $11\frac{1}{7}$ inches.

3. A cylindrical straight lever is 10 feet long, and weighs 6*lbs.*; its longer arm is 7, and its shorter 3 feet; at the extremity of its shorter arm a weight of 21 pounds is suspended; What weight must be placed at the extremity of the longer arm, to keep it in equilibrio? (See Art. 5, page 234.)

Ans. $7\frac{2}{7}$ pounds.

4. A

4. A body weighs 10*lbs.* 9*oz.* at one end of a false balance, and $12\frac{1}{4}$*lbs.* at the other; What is its *real* weight? Ans. 11*lbs.* and 6*oz.*

5. *A* and *B* are of the same height, and sustain upon their shoulders a weight of 150*lbs.* placed on a pole $8\frac{1}{2}$ feet long; the weight is placed $3\frac{1}{2}$ feet from *A*; What is the weight sustained by each person? See note (*a*), page 232.

Ans. *A* sustains $88\frac{4}{17}$*lbs,* and *B* sustains $61\frac{13}{17}$*lbs.*

6. The *longer* arm of a steelyard is 8 feet, and the *shorter* 6 inches; and its apparatus of *hooks* &c. is so contrived, that a weight of 5 pounds placed upon the longer arm, at the distance of one foot from the center of motion, will balance 12 pounds placed at the extremity of the shorter arm; the moveable weight (of 5*lbs.*) cannot conveniently be placed nearer to the fulcrum than 6 inches; What must be the *graduation* of the steelyard that it may weigh *half pounds,* and what will be the greatest and least weights that can be ascertained by it?

Ans. The *graduation* is $\frac{3}{5}$ of an inch; *greatest* W^t.$=82$*lbs,* *least*$=7$*lbs.*

7. The arms of a straight lever are to each other as 3 : 2, and it is acted upon *obliquely* by two forces; the force (*P*) applied at the extremity of the *longer* arm is inclined to it at an $\angle$ of 45°, and (*p*) at the *shorter*

shorter in an $\angle$ of 60°; What is the proportion between the forces, when the lever is in equilibrio? See Art. 3, page 237. Ans. $P : p :: \sqrt{6} : 3$.

8. The arms of a bent lever are *equal*, and $P : W :: 1 : \sqrt{2}$; What must be the inclination of the arms to each other, that the arm from whose extremity P is suspended may be *parallel to the horizon?*

Ans. 135°.

9. In a combination of levers connected together in the manner represented in the Fig. page 239, the three *shorter* arms (BC, DF, EG) are respectively 1, 2, and 4 feet; the three *longer* arms (AC, BF, DG) are 9, 11, and 12 feet; the weight (P) suspended from A is 3 pounds; What weight will it sustain at E? Ans. $445\frac{1}{2}$ *lbs.*

Lecture

Lecture VIII.

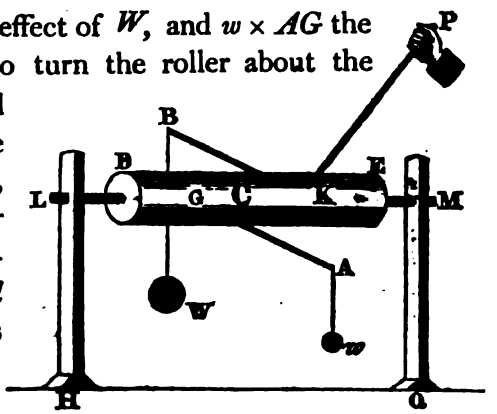
ON THE WHEEL AND AXLE; AND THE PULLEY.

THE principle of the *wheel and axle* is so analogous to that of the *lever*, that this may naturally be considered as the *next in order* of the mechanical powers.

XXXIII.

On the Wheel and Axle.

1. In order to explain the manner in which the wheel and axle operate upon each other, suppose DE to be a *cylindric roller* supported upon the props LH, MQ, and moveable about the axis LM. Let two straight inflexible rods AG, BC be inserted into this cylinder, in a direction *perpendicular to the axis*, but *parallel to each other and to the horizon;* let another rod PK be also inserted *perpendicular to the axis*, but making *any angle* with the plane passing through BC or AG. From the extremities of the rods BC, AG let the weights W, w be suspended; then (by Art. 2, page 230) $W \times BC$ will represent

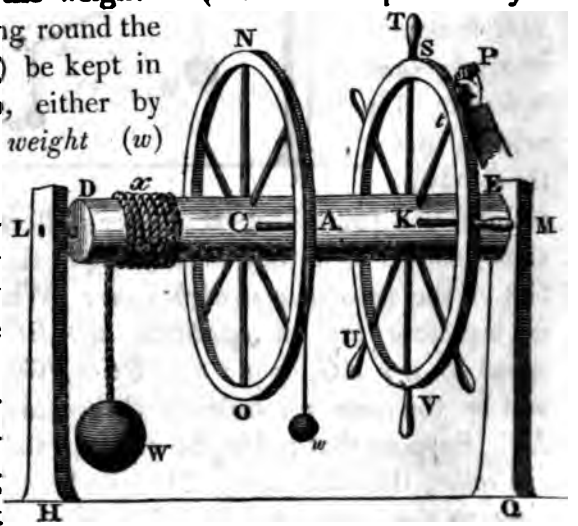


represent the effect of W, and $w \times AG$ the effect of w, to turn the roller about the axis LM; and supposing the *rods* AG, BC, and the *roller* DE, to be perfectly *rigid and inflexible*, it is evident that these two weights will counteract each other's effects, in the same manner as if they were acting at the extremities of the arms of a *straight lever*. When W and w therefore are in *equilibrio*, $W \times BC$ will be *equal* to $w \times AG$, or $w : W :: BC : AG$; and this will be the case *whatever be the length of the rod* BC. Suppose that rod to be equal to the *radius of the roller*, then the string BW will become a *tangent* to the roller, and the foregoing proportion becomes $w : W ::$ the *radius of the roller* $: AG$. Let us next suppose the weight W to be kept in equilibrio by a *power* P acting at right angles to the extremity of the rod PK; then may P and W be considered as acting at the extremities of the arms PK, BC of a *bent lever*; and since they act at *right angles* to those arms, (by Art. 3, page 237) $P : W :: BC : PK$[a]; and (when BW acts as a *tangent to the roller*) $P : W ::$ *radius of the roller* $: PK$.

2. This

(a) For in this case, sin. α = sin. β.

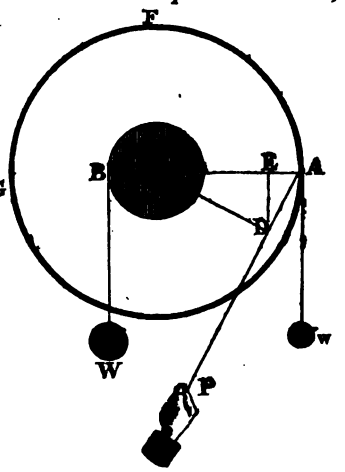
2. This mode of estimating the effects produced by forces acting upon a lever whose arms are in *different planes*, leads immediately to the explanation of the manner in which any weights or forces sustain each other by means of the *wheel and axle*. For, let the weight W (which is suspended by a rope going round the *axle* DE) be kept in equilibrio, either by *another weight* (w) suspended from a rope going freely round the wheel NAO, or by a *power* P acting at right



angles to the handles TS, Pt, &c. of the wheel StV, and let the planes of these wheels be at *right angles* to the axis LM of the machine; then, in the *former* case, the weight W may be considered as sustained by a weight (w) acting at right angles to the extremity of the arm AC of a *straight* lever, and in the *latter* by a power (P) acting at right angles to the extremity of the arm tK of a *bent* lever, the weight itself in each case acting at the distance of the *radius of the axle* from the center of motion; hence

hence, by Art. 1. w or $P : W ::$ radius of the axle $: AC$ or tK, i.e. "*the weight or power acting at the* "*circumference of the wheel : weight sustained upon* "*the axle :: radius of the axle : the radius of the* "*wheel*[a]."

3. If the power does not act at *right angles* to the radius of the wheel, but in some *oblique* direction, as AP (in the annexed Figure, which represents a *section* of the wheel and axle), then let fall CD perpendicular to AP. By the property of the *lever*, P and W are to each other "inversely as the perpendiculars let fall from "the center of motion "upon the directions in "which they respectively act;" in this case, therefore, $P : W :: CB : CD ::$ *radius of the axle : sine of the* $\angle$ *which* P *makes with the radius of the wheel.*



4. Hitherto we have not considered the *thickness of the rope;* when *that* is taken into the account, we

(a) Let $R =$ radius of the *wheel*, $r =$ radius of the *axle*, then $P : W :: r : R$, $\therefore P \times R = W \times r$; if W and r be *given*, and P and R *variable*, then $P \propto \frac{1}{R}$; i. e. "to sustain a given weight "upon a given axle, the power must be *increased* as the radius "of the wheel is *diminished*; and *vice versa*."

we must add the *half* of it to the distance at which W and w respectively act[a]. Let therefore $2t$ = *diameter of the rope*, and let R = rad. of the *wheel*, r = rad. *of axle*; then if the thickness of the rope be taken into the consideration, we have $w : W :: r+t : R+t$; and since in this case the *same* quantity (t) is added to each term of the ratio $r : R$[b], w must bear a *greater* ratio to W than that of $r : R$, or of the *radius of the axle* to the *radius of the wheel*[c].

5. To find the *pressure* upon the props LH, MQ (Fig. in p. 254), when two weights W, w keep themselves in equilibrio about the *wheel* NRO and *axle* DE,

(a) For W, w act in the direction of the *axis of the rope*, and this axis is evidently removed from the circumference of the wheel or axle by *half the thickness of the rope*.

(b) For $r : R$ is a ratio of *lesser inequality*, $\therefore$ it is *increased* by this addition of the *same* quantity to each of its terms. (Alg. Sect. 53. Lect. 8.)

(c) In this article we have considered the ropes which go round the wheel and axle to be of the *same* thickness, and that it coils round the axle but *once*. But suppose the thickness of the rope to which W is appended to be $2T$, that of w to be $2t$, and that the rope coils round the axle any number of times denoted by n, then it is evident, that for each coil of the rope after the *first*, W will be further removed from the circumference of the axle by the *whole* thickness ($2T$) of the rope; the *most general* form therefore under which the relation of $w : W$ can be exhibited, when *the thickness of the rope is taken into the consideration*, is, $w : W :: r+\overline{2n-1}.T : R+t$.

DE, let a = w^t. of the axle, and A = w^t. of the *wheel* NRO, and (in the annexed Figure) let X denote the distance of the weight (W) from L, and C the position of the *wheel*; moreover, let G be the *common center of gravity of* W *and* w;[d] then the *pressure of the axle* upon *each* prop $= \frac{1}{2}a$;[e] pressure of the weights W, w upon the *prop* $LH = \frac{\overline{W+w}.MG}{LM}$, and upon the *prop* $MQ = \frac{\overline{W+w}.LG}{LM}$; the pressure of the *wheel* upon $LH = \frac{A \times MC}{LM}$, upon $MQ = \frac{A \times LC}{LM}$; hence the *whole* pressure upon the prop $LH = \frac{1}{2}a + \frac{A \times MC + \overline{W+w}.MG}{LM}$, and upon the prop $MQ = \frac{1}{2}a + \frac{A \times LC + \overline{W+w}.LG}{LM}$.[f]

L X G C M

6. In

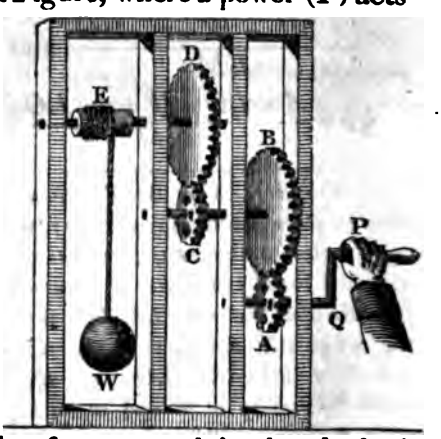
(d) The point X in this Figure is supposed to be that point where a plane passing through the rope xW (Fig. in pag. 254), at right-angles to the axis, *intersects* the axis; and the point C is supposed to be the *center of the wheel*; so far therefore as the *pressure upon the props* LH, MQ is concerned, it is evident that the weights W, w may be considered as placed at X and C respectively. Take $GX : GC :: w : W$, then G will be the *common center of gravity* of W and w, and the pressure of W and w upon the props will be the same as if a weight equal to $W+w$ was placed at G. (Art. 7. pp. 99, 100.)

(e) For the *weight of the axle* acts in a similar manner upon *each* of the props; and as the two props sustain its *whole* weight, it is evident that each prop must sustain *half* that weight.

(f) For the manner of calculating the respective pressures of A and $W+w$ upon the props LH, MQ, the reader is referred to Note (*a*), page 232, and Note (*a*), page 234.

6. In the same manner, we might compare the pressure upon the props LH, MQ, when the weight (W) is sustained by means of a power (P) acting upon the *handles* of the wheel StV (Fig. in page 254); except that in this case, since P and W may act on the *same* or on *opposite* sides of the axis of motion, and since P acts at different degrees of *obliquity* during one revolution of the wheel, that part of the expression which involves the pressure of P and W will be *variable*, and assume the form of $mW \pm nP$, where the *positive* or *negative* sign must be used according as W and P are on *opposite* sides or on the *same* side of the axis.

7. In a *combination* of wheels, such as is represented in the annexed Figure, where a power (P) acts upon the winch or handle PQ,[a] which turns the wheel A, which acts upon the wheel B, from which the motion is propagated through the wheels C and D to the axle E, about which the rope that sustains the weight (W) is wound, let the *force exerted by the wheel A upon*



(a) In this case the effect will evidently be the same as if the power acted *at the circumference of a wheel whose radius is PQ*.

DE, let a = w^t. of the axle, and A = w^t. of the *wheel* NRO, and (in the annexed Figure) let X denote the distance of the weight (W) from L, and C the position of the *wheel*; moreover, let G be the *common center of gravity of* W *and* w;[d] then the *pressure of the axle* upon *each* prop = $\frac{1}{2}a$;[e] pressure of the weights W, w upon the *prop* $LH = \frac{\overline{W+w}.MG}{LM}$, and upon the *prop* $MQ = \frac{\overline{W+w}.LG}{LM}$; the pressure of the *wheel* upon $LH = \frac{A \times MC}{LM}$, upon $MQ = \frac{A \times LC}{LM}$; hence the *whole* pressure upon the prop $LH = \frac{1}{2}a + \frac{A \times MC + \overline{W+w}.MG}{LM}$, and upon the prop $MQ = \frac{1}{2}a + \frac{A \times LC + \overline{W+w}.LG}{LM}$[f].

L X G C M

6. In

(d) The point X in this Figure is supposed to be that point where a plane passing through the rope xW (Fig. in pag. 254), at right-angles to the axis, *intersects* the axis; and the point C is supposed to be the *center of the wheel*; so far therefore as the *pressure upon the props* LH, MQ is concerned, it is evident that the weights W, w may be considered as placed at X and C respectively. Take $GX : GC :: w : W$, then G will be the *common center of gravity* of W and w, and the pressure of W and w upon the props will be the same as if a weight equal to $W+w$ was placed at G. (Art. 7. pp. 99, 100.)

(e) For the *weight of the axle* acts in a similar manner upon *each* of the props; and as the two props sustain its *whole* weight, it is evident that each prop must sustain *half* that weight.

(f) For the manner of calculating the respective pressures of A and $W+w$ upon the props LH, MQ, the reader is referred to Note (*a*), page 232, and Note (*a*), page 234.

Let r = radius of *axle*, x = ... of wheel, then, by Art. 2,
$P\,(12) : W\,(100) :: r\,(6) : x$,
$\therefore\ x = \frac{600}{12} = 50$ *inches* = Ft. 4 In. 2,
and the *diameter* = 8 4.

EXAM. II.

A weight of 500 *lbs. is sustained by a rope of one inch diameter, going round an axle whose radius is* 8 *inches; and the power acts close to the circumference of a wheel whose radius is* 4 *feet; what is the ratio of* $P : W$?

This is a case of Art. 4, where the weight is not kept in equilibrio by *another weight*, but by some *power* acting upon a handle close to the circumference of the wheel (as P acts upon the wheel StV in Fig. page 254), and $\therefore\ t$ disappears in the 4th term of the proportion w (or P) : W :: $r + t$: $R + t$, which becomes $P : W :: r + t : R$. In the present instance, $W = 500$, $r = 8$ inches, $t = \frac{1}{2}$ inch, $R = 48$ inches, $\therefore\ P : 500 :: 8\frac{1}{2} : 48$, or $P = \frac{4250}{48}$ $= 88.54$ *lbs.*[a]

EXAM. III.

In the Figure, page 258, PQ = *one foot; radii of wheels A, C are each* 4 *inches; radii of wheels B, D are*

(a) If the *thickness of the rope* be not considered, then $P : W$ (500) :: r (8) : R (48), $\therefore\ P = \frac{4000}{48} = 83.33$; it makes a *difference* therefore of 5.21 lbs. whether this thickness be or be *not* taken into the account.

are each 15 *inches; and the radius of the axle E is* 3 *inches; what power must be applied at P to support a weight of* 600*lbs.?*

By Art. 7, $P : W$ (600) :: $a \times r \times r' : PQ \times R \times R'$,

:: $3 \times 4 \times 4 : 12 \times 15 \times 15$,

:: $4 : 225$, $\therefore P = \frac{2400}{225} = 10\frac{2}{3}$ *lbs.*

EXAM. IV.

In a combination of wheels and axles, in which the circumference of each axle is applied to the circumference of the next wheel, and in which the ratios of the radii of the wheels and axles are 2 : 1, 4 : 1, 8 : 1, &c...., $2^n : 1$ (*where* n = *the number of wheels*) *there is an equilibrium when the power : weight* :: 1 : p. *Required the number of wheels?*

This is a case of Art. 8, where $P : W :: r \times r' \times r''$ &c. : $R \times R' \times R''$ &c; and in the present instance, $1 : p :: 1 \times 1 \times 1$ &c. : $2 \times 2^2 \times 2^3$ &c...2^n :: $1 : 2^{1+2+3+\&c...n} :: 1 : 2^{n \cdot \frac{n+1}{2}}$, $\therefore p = 2^{n \cdot \frac{n+1}{2}}$, and $p^2 = 2^{n^2+n}$; hence 2 log. $p = n^2 + n$.log. 2, or $n^2 + n = \frac{2 \log. p}{\log. 2}$ (which put) $= a$; then solving the quadratic equation, we have $n = \frac{\sqrt{4a+1} - 1}{2}$.

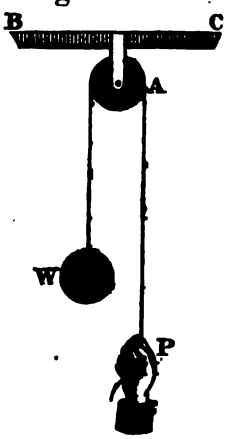
XXXIV.

On the Pulley, and System of Pulleys.

A *pulley* is a small *grooved* wheel moveable about a pivot, the pivot itself being at the same time either *fixt* or *moveable*. The principle upon which a weight is

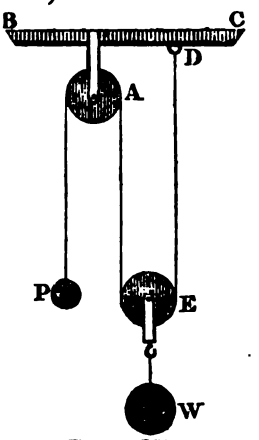
is sustained by means of a pulley or *system* of pulleys is very simple, and will be readily understood from the following investigation.

1. In the *single fixt* pulley (*A*), about which the weight (*W*) is sustained by the power (*P*) acting on a string (*WAP*) passing along the groove in the circumference, no *mechanical advantage* is gained; for since the rope passes *freely* round the pulley, it is evident that the tension on each side of it must be the same, and consequently that the power must be *equal* to the weight which it sustains. The only advantageous circumstance attending a pulley of this kind is, that a given power may be made to sustain or put in motion a given weight in a more *convenient* manner, by altering at pleasure the direction in which the power acts. The *pressure* upon the pivot or axis of the pulley (*A*) is evidently equal to $P + W$.

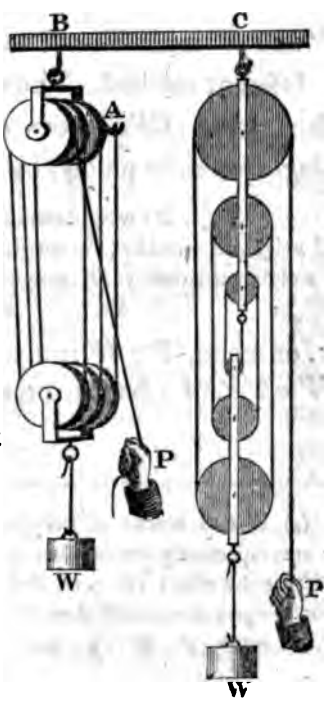


2. But if a weight (*W*) be sustained by a power (*P*) acting on a string going over a *moveable* pulley (*E*) as well as the *fixt* one *A*, then it is evident that this weight is sustained by the *two* strings *AE*, *DE*; and as it is suspended from the *center* of the pulley (*E*), these ropes must act at *equal* distances from that center; consequently, each string must sustain *half*

half the weight (Note, page 232). But it is evident, that whatever is the weight sustained by the string *AE*, the *same* must be sustained by the power (*P*), which acts upon a string going freely over the fixt pulley (*A*); hence, when there is an equilibrium, $P = \frac{1}{2}W$, or $W = 2P$, $\therefore P : W :: 1 : 2$. With respect to the *pressure* upon the hook *D*, it is $\frac{1}{2}W$ or *P*, and upon the axis of the pulley (*A*) it is equal to $P + \frac{1}{2}W = 2P$.

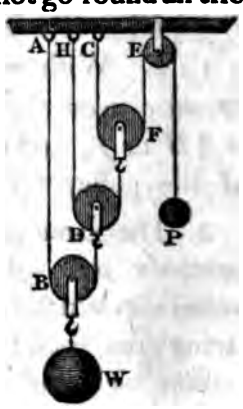


3. The same principle pervades the *system* of pulleys, in which the *same* string goes round *all* the pulleys, as in the annexed Figures. For it is evident that the weight (*W*) is supported by *all* the strings at the *lower* block; if therefore the *number* of these strings be (*n*), each string must support $\frac{1}{n}$th part of the weight. But when there is an equilibrium, whatever be the tension upon each of these strings which support the weight, the *same* will be the tension



on the string upon which the power (P) acts; hence $P = \frac{1}{n} W$, or $W = nP$; $\therefore P : W :: 1 : n$ (a) (where n = number of strings at lower block, or *twice* the number of *moveable* pulleys). The *pressure upon the hook B or C* is evidently equal to $P + W = P + nP = \overline{n+1}.P$.

4. When the *same* string does not go round all the pulleys, but each pulley has a *separate* string *CFE*, *HDF*, *ABD*, &c. going round it, and fastened to the hooks *A*, *H*, *C*, &c. then the relation between *P* and *W* must be estimated by a different method. Thus (since the string *CFE* goes over a *single moveable* pulley) by Art. 2,



P : weight sustained by pulley (F) :: 1 : 2;
and weight sustained by (F) : weight sustained by (D) :: 1 : 2,
weight sustained by (D) : weight sustained by (B) i.e. W :: 1 : 2,
&c. &c. &c.

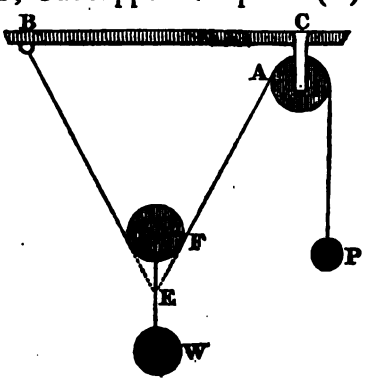
$\therefore$ ex æquo, $P : W :: 1 : 2 \times 2 \times 2$, &c. $:: 1 : 2^n$, or $W = 2^n P$ (if n be the number of *moveable* pulleys).

In

(a) If *two* blocks of pulleys of this kind (in which m and n are respectively the number of strings) were combined together, so that the effect (E) produced by the *first* block should act as *power* upon the *second*, then $P : E :: 1 : m$, and $E : W :: 1 : n$; $\therefore$ ex æquo, $P : W :: 1 : mn$.

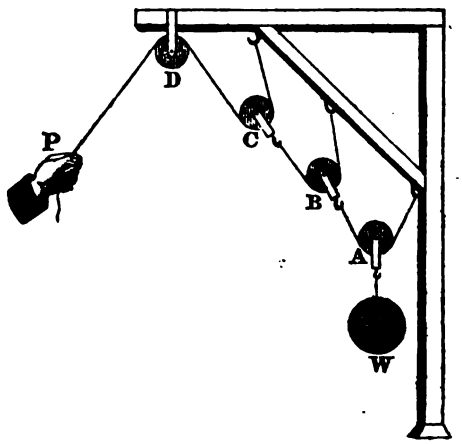
In this system of pulleys, the *pressure* upon the hook $A = \frac{1}{2}W = \frac{1}{2} \times 2^n P = 2^{n-1}P$; upon the hook H ($= \frac{1}{2}$ pressure upon A) $= \frac{1}{2} \times 2^{n-1}P = 2^{n-2}P$; &c. &c. and the pressure upon pulley $E = 2P$.

5. Hitherto we have considered the strings as acting *parallel* to each other; but suppose the power (P) to act upon the weight (W) by a string going over the moveable pulley (E) in an *oblique* direction, then produce the string AF to E, and draw DF at right angles to DE (D being the *center* of the pulley). Let FE represent the magnitude of the power acting in the direction EF, which resolve into ED, DF; then ED is that part of it which is efficacious in supporting the weight (W); and since the string BD supports the same weight which the string AF does, the whole weight sustained by the string BFA will be represented by $2DE$; hence $P : W :: EF : 2DE :: rad. : 2 \cos. \angle DEF$ or $2 \cos. \angle$ *of inclination of the direction of the power to the direction of the weight.*



6. Suppose there to be a system of pulleys (of the same nature with that in Art. 4.) in which the strings do not act *parallel* to each other, but the string CD is

is inclined to the string CB in an $\angle(\alpha)$; BC inclined to BA in $\angle(\beta)$; BA to AW in $\angle(\gamma)$ &c.; then, from what has just been proved,

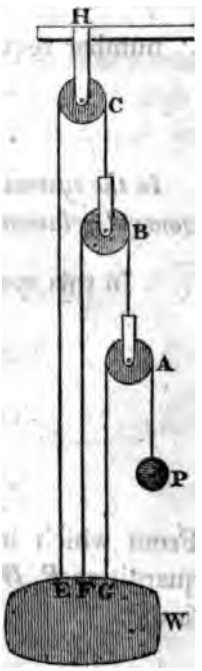


P : weight sustained at C :: rad. (1) : 2 cos. α,
weight at C : weight at B :: rad. (1) : 2 cos. β,
weight at B : weight at A (W) :: rad. (1) : 2 cos. γ,
&c. &c. &c. &c.

$\therefore$ ex æquo, $P : W :: 1 : 2^n \times \cos.\alpha \times \cos.\beta \times \cos.\gamma$ &c., where n = the number of *moveable* pulleys.

7. There is another mode of combining pulleys together, which we have not yet noticed; viz. when *each string is fixt into the weight*, as in the annexed Figure. In this case, supposing there to be an equilibrium, and the power (P) to act upon a string going freely over the pulley A, then it is evident that the pressure upon that pulley will be equal to $2P$, $\therefore$ the string BA supports a part of the weight equal

equal to $2P$. For the same reason, since the string FBA goes freely over the pulley B, the string CB supports $4P$; &c. &c. hence the portions of weight supported by the strings AG, BF, CE, &c. are $P, 2P, 4P$, &c. respectively, and consequently $W = P + 2P + 4P$ + &c. . . . $2^{n-1}P$ (where n = number of strings attached to the weight) $= P \cdot \overline{1 + 2 + 4 + \&c.2^{n-1}}$ $= P \cdot \overline{2^n - 1}$,(a) $\therefore P : W :: 1 : 2^n - 1$. The *pressure* upon the support at H is evidently equal to $P + W = P + \overline{2^n - 1} \cdot P = 2^n P$.



EXAM. I.

A weight of 56 *lbs. is kept in equilibrio by a power of* 7 *lbs. by means of a system of pulleys, in which the same string goes round every pulley; What is the number of moveable pulleys?*

Let n = the number of strings at the *lower* block of pulleys; then, by Art. 3, P (7) : W (56) :: 1 : n =

(a) The sum of a *geometric* series, whose *first term* is 1, *common ratio* 2, and *number of terms* n, is $2^n - 1$. (See Alg. Lect. 12. Sect. 73.)

$= \frac{56}{7} = 8 =$ *twice* the number of moveable pulleys,

$\therefore$ number required is 4.

EXAM. II.

In the system of pulleys described in Art. 4. *find the general relation between* P, W, *and* n.

In this system, $W = 2^n P$, $\therefore P = \frac{W}{2^n}$;

and $2^n = \frac{W}{P}$,

$\therefore n \cdot \log. 2 = \log. W - \log. P$,

or $n = \frac{\log. W - \log. P}{\log. 2.}$.

From which it appears, that if any two of the three quantities P, W, or n, be given, the *third* may be found.

EXAM. III.

In the system of pulleys described in Art. 6, *where the strings act* obliquely *upon each pulley, the angles* α, β, γ, &c. *are each of them* 60°; *What is the relation between P and W?*

In this system, $P : W :: 1 : 2^n \times \cos. \alpha \times \cos. \beta \times \cos. \gamma$, &c.; hence, as α, β, γ, &c. are each of them 60°, cos. α, cos. β, cos. γ, &c. are each of them equal to (*sin.* 30° or) $\frac{1}{2}$, and the number of these quantities is n; $\therefore P : W :: 1 : 2^n \times \frac{1}{2^n} :: 1 : 1$; or $P = W$, *whatever be the number of pulleys.*

Ex.

EXAM. IV.

Find the general relation between P, W, *and* n, *in the system of pulleys described in Art.* 7; *and then find the number of pulleys necessary for a power of* 3 *lbs. to support a weight of* 381 *lbs.*

Here $W=P.\overline{2^n-1}$, $\therefore P=\frac{W}{2^n-1}$;

also, $2^n-1=\frac{W}{P}$, $\therefore 2^n=\frac{W}{P}+1=\frac{W+P}{P}$;

hence $n\times\log.2=\log.\overline{W+P}-\log.P$, or $n=\frac{\log.\overline{W+P}-\log.P}{\log.2}$.

If $W=381$, and $P=3$; then $n=\frac{\log.384-\log.3}{\log.2}=\frac{2:107210}{0.301030}=7$.

XXXV.

QUESTIONS FOR PRACTICE.

1. A power of 3lbs. acts upon a wheel whose diameter is 6 feet; What weight will it keep in equilibrio, supposing the rope which supports that weight to be wound round an axle whose diameter is 5 inches? Ans. 43$\frac{1}{2}$ *lbs.*

2. A power of 5lbs. keeps in equilibrio a weight of 150lbs. by means of a wheel whose diameter is 10 feet; What is the diameter of the *axle?*

Ans. 4 inches.

3. A power (P) acting by means of rope going over a wheel whose diameter is 6 feet, supports a weight

a weight of 320lbs; the diameter of the *axle* is 8 inches, and the rope by which P and W are suspended is 2 inches thick; What must be the magnitude of P, supposing the *thickness of the rope* to be taken into consideration?

Ans. $P = 43.24$ *lbs.*

4. Four wheels, A, B, C, D, whose diameters are 5, 4, 3, and 2 feet respectively, are put in motion by a power of 10lbs. applied at the circumference of the wheel A; these wheels act upon each other by means of three *smaller* wheels, the diameter of each of which is 8 inches; the *last* wheel D turns an axle whose diameter is 6 inches; What weight may be sustained by a rope going over this axle? Ans. 8100 *lbs.*

5. In the combination of wheels and axles described in Exam. 4. p. 261, $P : W :: 1 : 32768$;(*) What is the number of wheels?

Ans. 5 wheels.

6. By means of a system of pulleys, of which 6 are *moveable*, and the same string goes round *all* the pulleys, what power will be necessary to sustain a weight of 112lbs? Ans. 9$\frac{1}{3}$ *lbs?*

7. A

(*) The logarithm of this number, as it stands in the Tables, is 4.5154499; but, for the solution of this question, it will be proper to make it 4.5154500.

7. A weight is sustained by a power attached to a rope going over *one* moveable Pulley (as in Fig. page 265), the direction of the rope making an $\angle$ of 45° with a verticle line passing through the center of the pulley; What is the relation between P and W? Ans. $P : W :: 1 : \sqrt{2}$.

8. A weight of 640lbs. is sustained by a power equivalent to 5lbs. by means of the system of pulleys described in Art. 4, p. 264; What is the number of pulleys? Ans. 7 pulleys.

9. What power will be necessary to sustain a weight of 1020lbs. in a system of 8 pulleys, constructed according to the Fig. in p. 267, where the strings are all fastened to the weight?

Ans. 4lbs.

Lecture IX.

ON THE INCLINED PLANE, THE SCREW, AND THE WEDGE.

THIS Lecture will comprehend the three remaining mechanical powers; viz. the *Inclined Plane*, the *Screw*, and the *Wedge*; beginning with the *Inclined Plane*, being that upon which the properties of the *Screw* more immediately depend.

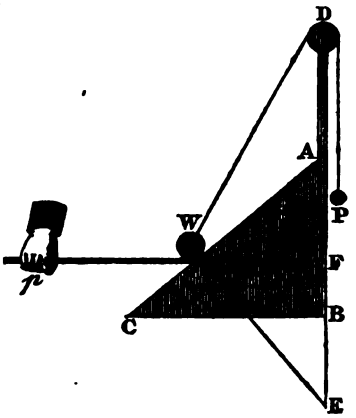
XXXVI.

On the Inclined Plane.

1. Let *AC* be an inclined plane, whose *length* is *AC*, *height AB*, and *base BC*; and suppose the *weight W* to be kept in equilibrio by *another weight* (or power) *P* acting freely over a pulley fixt at *D*. Draw[a] *WE* at right angles to *AC*, meeting *AB*

(a) This Figure is in course to be considered as a *section* of the plane passing through the *center of gravity of the weight*; and if the weight be not large, that center of gravity may be considered as placed in the angular point (*W*) of the triangle *DWE*.

AB (produced if necessary) in *E*; the weight *W* may then be considered as kept at *rest* by three forces, viz. the action of the *power* in the direction *WD*, its own weight (or *gravity*) in direction *DE*, and the *re-action* of the plane in direction *EW*; ∴ (by Art. 2, page 70) these three forces are to each other as the three sides of the triangle *DWE*, in the directions of which they respectively act. Hence *P* : *W* :: *WD* : *DE* :: sin. ∠*WED* or ∠*ACB*[b] : sin. ∠*DWE* :: sine of the ∠ of *inclination of the plane* : sine of the ∠ which *the direction of the power makes with a perpendicular to the plane at the point where the weight rests upon it.*



2. By the *third* law of motion, the *pressure* of *W* upon the plane must be equal to the *re-action* of the plane upon *W*; if therefore *EW* represents that re-action, *WE* will represent the *pressure* upon the plane; hence *P* : *pressure upon the plane* :: *WD* : *WE* :: sin. ∠*WED* or ∠*ACB* : sin. ∠*WDE*; and *W* : *pressure upon the plane* :: *DE* : *WE* : sin. ∠*DWE* : sin. ∠*DWE*.

3. If

(b) Since the triangles *ACB*, *WAE*, are *right-angled* triangles, and have the angle at *A common*, the angle *AEW* must be equal to the angle *ACB*.

3. If the power acts *parallel to the plane*, then WD may be considered as coinciding with WA, and the *power*, the *weight*, and the *pressure* will be respectively represented by the three sides WA, AE, WE of the triangle AWE; hence $P : W :: WA : AE$:: (by sim. Δ') $AB : AC$:: *height* of the plane : *length* of the plane; and P : *pressure upon the plane* :: $WA : WE :: AB : BC$:: *height* of the plane : *base* of the plane; also W : *pressure upon the plane* :: $AE : WE :: AC : BC$:: *length* of the plane : *base* of the plane.

4. If the power acts parallel to the *base* of the plane, (i. e. if the weight W be sustained upon the plane by a force acting in direction pW, and *pushing upwards* in such manner that pW is always parallel to the horizon) then produce pW to F; and when there is an equilibrium, the *power*, the *weight*, and the *pressure* will be respectively represented by the three sides WF, FE, WE of the triangle WFE; $\therefore P : W :: WF : FE$:: (by sim. Δ') $AB : BC$:: *height* of the plane : *base* of the plane; and P : *pressure upon the plane* :: $WF : WE :: AB : AC$:: *height* of the plane : *length* of the plane; also, W : *pressure upon the plane* :: $FE : WE :: BC : AC$:: *base* of the plane : *length* of the plane.

5. Let $\alpha = \angle$ of inclination of the plane, $\beta = \angle$ which the direction of the power makes with a perpendicular to the plane at the point where the weight rests upon it; then, by Art. 1, $P : W$:: sin. α : sin.

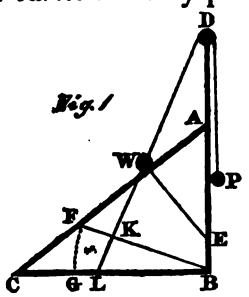
: sin. β; $\therefore P = \frac{W \times \text{sin. } \alpha}{\text{sin. } \beta}$. Suppose W and sin. α to be given, then P varies as $\frac{1}{\text{sin. } \beta}$, and will consequently be the *least* when sin. β is the greatest, i. e. when the angle DWE becomes a *right angle*, or P acts *parallel to the plane*; hence the *least* power will be required to raise or sustain a *given weight* upon a *given inclined plane*, when the *direction* in which that power acts is *parallel to the plane*. Again, $W = \frac{P \times \text{sin. } \beta}{\text{sin. } \alpha}$; if therefore P and *sin.* α be given, then $W \propto$ sin. β, and will consequently be *greatest* when sin. β is greatest, i. e. the *greatest weight* will also be raised or sustained by a *given power* upon a *given inclined plane*, when the direction in which the power acts is parallel to the plane.

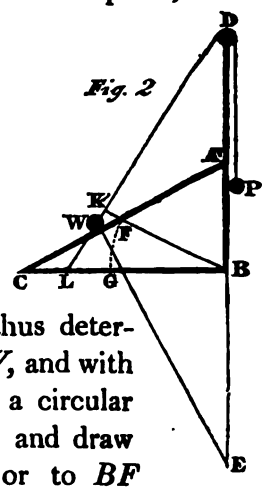
6. Let $\gamma = \angle WDE$; then, by Art. 2, P : *pressure upon the plane* :: sin. α : sin. γ, $\therefore$ *pressure upon the plane* $= \frac{P \times \text{sin. } \gamma}{\text{sin. } \alpha}$; if P and sin. α be *given*, then the pressure upon the plane $\propto$ sin. γ, and will consequently be *greatest* when sin. γ is greatest, i. e. when WDE becomes a *right angle*, or the power acts *parallel to the base of the plane*[a].

7. When

(a) In this case, since WE is greater than FE, it appears that the *pressure* is *greater than* W; but in this there is no inconsistency, for it is evident that when a weight is sustained upon an inclined plane by means of a force acting in direction p W, *part* of the pressure arises from the *power* as well as from the *weight*, and therefore the *whole* pressure *may* be greater than the weight.

7. When a weight (W) is sustained by another weight (P) going over a fixt pulley at D, since the angle DWE *varies* at every point of the plane, it is evident that there is but *one* point of the plane where a *given power* will sustain a *given weight;* and that point may be thus determined. Take $BG : BC :: P : W$, and with center B and radius BG describe a circular arc cutting AC in F; join BF, and draw DKL perpendicular to BF, or to BF *produced;* then the point W where DL cuts AC will be the point required; for the triangles DWE, BFC being similar,(a) we have $WD : DE :: BF$ or $BG : BC$; but $P : W :: BG : BC$, $\therefore P : W :: WD : DE$; hence, by Art. 1, the relation between the sides of the triangle WDE is such as to give the position of W when there is an equilibrium between P and W.





8. But when the power acts *parallel to the plane,* or *parallel to the base of the plane,* the ratio of $P : W$

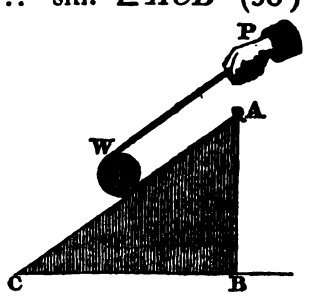
(a) Draw WE at right angles to AC, as before, then $\angle AEW = \angle ACB$; and in the *right-angled* triangles BKL, DBL, the $\angle KLB$ is *common*, $\therefore \angle KBL = \angle KDB$; hence the $\angle CFB$ =the $\angle DWE$, and consequently the triangle CFB is *similar* to the triangle DWE.

$P : W$ is *constant*[b] through every part of the plane; if therefore a given power be in equilibrio with a given weight at any *one point* of a plane, the same power would also be in equilibrio with the weight, when placed at *any other point* of that plane. We now proceed to give a few examples, for the illustration of this theory.

EXAM. I.

A person is just able to sustain by his strength a weight of 200 *lbs. What weight would he be able to sustain on an inclined plane whose elevation is* 50°, *by means of a rope going round it, and fixt to the top of the plane in the manner represented in the annexed Figure?*

In this case, the power which supports the weight acts *parallel to the plane*, ∴ by Art. 3, $P : W :: AB : AC ::$ sin. $\angle ACB$ (50°) : radius :: 7660 : 10000,



or $W = \frac{P \times 10000}{7660}$; this is the weight supported by the rope PWA; but since that rope is *fixt* at A, each part PW, WA of that rope supports *half*[c] the weight; hence, if the force exerted

(b) For $P : W$ (in *one* case) ∴ height of the plane : length of plane; and in the *other*, $P : W ::$ height of plane : base of plane (Art. 3, 4.)

(c) The case being similar to that of a weight supported by two *parallel* strings going over a pulley.

exerted by the rope $PWA = P$, the force acting in the direction $WP = \frac{1}{2}P$; call that force (p), then $p = \frac{1}{2}P$, or $P = 2p$; substitute this for P, and we have $W = \frac{2p \times 10000}{7660} = (\text{if } p = 200)\ \frac{400000}{766}$ $= 522.19 lbs.$; if therefore a person by his natural strength is able to lift a weight of $200 lbs$, acting under the circumstances here represented he will be able to sustain a weight of $522.19 lbs$.

EXAM. II.

Upon an inclined plane, whose ~~height~~ length is 20 feet, and elevation 30°, a weight of 3lbs. is sustained by a power of 2lbs, acting over a pulley fixt at the distance of 10 feet from the top of the plane (in the manner represented in Fig. 1, page 276); It is required to find the distance of W from the top of the plane, when there is an equilibrium.

Since $AC = 20$ feet, and $\angle ACB = 30°$, AB (= sin. 30° to radius AC) will be equal to 10 feet, and consequently $BC = \sqrt{AC^2 - AB^2} = \sqrt{300} = 17.32$ feet; constructing the Figure, therefore, as in Art. 7, we have $BG : BC\ (17.32) :: P\ (2) : W\ (3)$, $\therefore$ BG (or BF) $= \frac{2 \times 17.32}{3} = 11.54$ feet; hence, in the triangle BFC we have $BC = 17.22$, $BF = 11.54$, $\angle FCB = 30°$, from which the angles FBC, CFB are found to be respectively 18°.37′, and 131°.23′; but the triangle DWE is similar to the triangle BFC, therefore

$\therefore$ $\angle WED = 30^\circ$, $\angle WDE = 18^\circ.37'$, and $\angle DWE = 131^\circ.23'$.

Again, since DWE is $131^\circ.23'$, and AWE a right $\angle$ (or 90°), the $\angle DWA$ must be $41^\circ.23'$; hence, in the triangle DWA, we have $AD = 10$ feet, $\angle WDA = 18^\circ.37'$, and $\angle DWA = 41^\circ.23'$, from which AW is found to be 4.828 feet, which gives *the distance of W from the top of the plane when P is in equilibrio with W.*

EXAM. III.

A body is sustained upon an inclined plane, first by a power acting parallel to the plane, and afterwards by a power acting parallel to the base of the plane. Compare the pressure upon the plane in these two different cases.

By Art. 3, when the power acts *parallel to the plane*, *W* : *pressure upon the plane* (*P*) :: *length* of the plane : *base* of the plane.

By Art. 4, when it acts *parallel to the base of the plane*, *Pressure upon the plane* (*p*) : *W* :: *length* of the plane : *base* of the plane.

Hence, *ex æquo*, *p* : *P* :: the *square* of the *length* of the plane : *square* of the *base* of the plane. Thus, in a plane whose elevation is 60° (and whose length is consequently *double*[a] of its base), it makes a difference

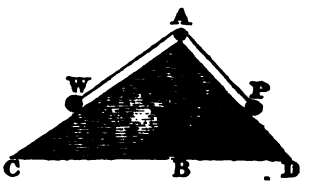
(a) In Fig. page 273, if $\angle ACB = 60^\circ$, then $\angle CAB = 30^\circ$; $\therefore$ $BC = \frac{1}{2}$ *radius* $= \frac{1}{2}AC$, or $AC = 2BC$.

difference of 4 : 1 as to the *pressure upon the plane*, whether a body be sustained upon it by a force acting *parallel to the plane*, or by one acting *parallel to the base of the plane.*

EXAM. IV.

Two weights P, W resting upon the inclined planes AC, AD, whose common height is AB, keep each other in equilibrio by means of a string going over a pulley fixt at A. Compare the two weights.

Since the string passes *freely* over the pulley at *A*, and the two weights are *at rest*, it is evident that the tension of the string *WAP* must be every where the same, i.e. whatever power is exerted at *A* to sustain *W* on the *plane AC*, the *same* is exerted at that point to sustain *P* upon the *plane AD*; call that power (*p*), then, since the power in each case may be considered as acting *parallel to the plane*, we have, by Art. 3,



$$p : W :: AB : AC,$$
$$\text{and } P : p :: AD : AB;$$

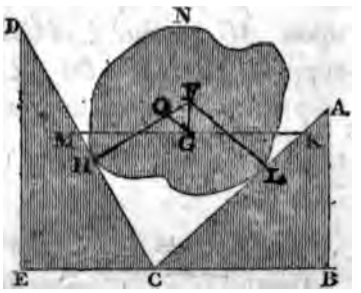
$\therefore P : W :: AD : AC ::$ plane upon which *P rests* : plane upon which *W* rests.

EXAM. V.

A body is supported between two inclined planes of given elevations; compare the pressure upon the planes.

Let

Let *NHL* represent a perpendicular *section* of the body passing through its two points of contact (*H, L*) with the planes; and from those points draw *HF, LF* at right angles to the planes *DC, AC* respectively. From their intersection *F*, draw *FG* perpendicular to the horizon, and let it represent the *weight of the body*. Through *G* draw *GO* parallel to *LF*; then the three sides of the triangle *GOF* will be in the direction of the three forces which keep the body at *rest* upon the plane *AC*, viz. *GO* will represent the re-action of the *plane AC*, *OF* the re-action of the *plane DC*, and *FG* the *weight of the body*; and in the same manner it may be shewn that the three sides of the same triangle will represent the three forces which keep the body at rest upon the plane *DC*. Through *G* draw *MGK* parallel to the horizon; then since the three sides of the $\triangle GOF$ are *perpendicular* to the three sides of the triangle *MCL*(a), *MCK* must be *similar* to *GOF* (see Note, page 71); and since the *weight* of the *body*, the *pressure upon the plane DC*, and the *pressure upon the plane AC*, are respectively represented by the three sides *FG, FO, OG* of the $\triangle GOF$, they will *also* be represented by the



(a) For *FG* is perpendicular to *MK*, *FO* to *HC*, and *OG* (being *parallel* to *FL*) is *perpendicular* to *KC*.

the three sides MK, MC, CK, of the triangle MCL. Hence the *pressure upon* DC : *weight of the body* :: (FO : FG ::) CM : MK :: sin. $\angle MKC$ or ACB : sin. $\angle MCK$; and the *weight of the body* : the *pressure upon* AC :: (FG : OG ::) MK : CK :: sin. $\angle CMK$ or DCE : sin. $\angle MCK$; $\therefore$ *ex æquo*, the pressure upon DC : the pressure upon AC :: sin. $\angle ACB$: sin. $\angle DCE$. Thus suppose $\angle DCE = 60°$, $\angle ACB = 30°$, then pressure on DC : pressure on AC :: sin. 30° : sin. 60° :: $\frac{1}{2} : \sqrt{\frac{3}{4}} :: 1 : \sqrt{3}$.

Cor. Let $\alpha = \angle DCE$, $\beta = \angle ACB$, then $\angle MCK = 180° - \overline{\alpha + \beta}$; $\therefore$ pressure on DC : W :: sin. β : sin. $\overline{180° - \alpha + \beta}$, and pressure upon AC : W :: sin. α : sin. $\overline{180° - \alpha + \beta}$; consequently the *sum* of the pressures upon DC, AC : W :: sin. α + sin. β : sin. $\overline{180° - \alpha + \beta}$. Thus, if each of the planes be inclined to the horizon in an $\angle$ of 60°, then *sum of the pressures* : W :: 2 sin. 60° : sin. 60° :: 2 : 1, i. e. the *sum of the pressures = twice the weight of the body*.

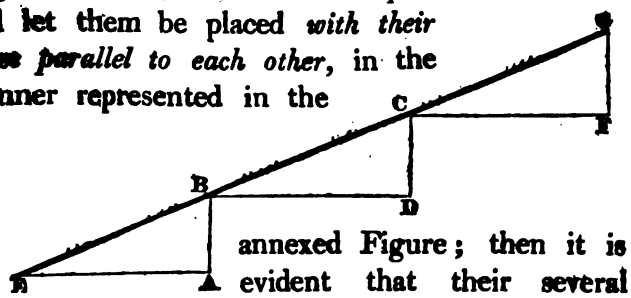
XXXVII.

On the Screw.

The *Screw* combines the principle of the *Inclined Plane* with that of the *Lever*. In order to explain the nature of this machine, it will therefore be necessary to

to resolve it into its constituent parts, and describe the manner in which these parts act severally upon each other.

1. Let AEB, DBC, FCG, &c. be a series of inclined planes, whose *bases* EA, BD, CF, &c. and *heights* AB, DC, FG, &c. are all *equal* to each other; and let them be placed *with their bases parallel to each other*, in the manner represented in the



annexed Figure; then it is evident that their several *lengths* EB, BC, CG, &c. will be all equal, and that they will fall in *one right line* $EBCG$. From Art. 4, page 274, it appears that if a power acts *parallel to the base* of any one of these planes, its relation to the *weight* which it can sustain will be that of the *height* of the plane to the *base* of the plane; and it moreover appears, from Art. 8, page 277, that it is immaterial at what point of the plane the power acts; from which it follows that a weight (W) would be sustained by a power (P) acting, parallel to the horizon, at *any point* of the line $EBCG$, if $P : W ::$ *height of* any one of these planes : its *base*.

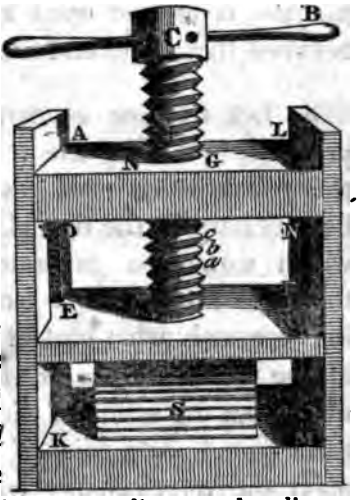
2. Suppose now the *preceding* Figure to be wound round the *cylinder* $AEGL$ (the circumference

rence of whose base is just equal to any one of the lines *AE*, *BD*, *CF*, &c.) so that the plane *EBCG* should form a *protuberant spiral line* *EHBKCLG* about the surface of the cylinder; each of the parts *EHB*, *BKC*, *CLG* of that spiral line would in this case be *similar and equal* to each other, and their *heights* *EB*, *BC*, *CG*, &c. would be equal to the heights *AB*, *DC*, *FG*, &c. of the several inclined planes *AEB*, *DBC*, *FCG*, &c. As the inclination of every part of this spiral line to the horizon is the same as that of the plane *EBCG*, it is evident that the power necessary to sustain a weight upon it will be the same as that upon the plane; viz. *P* (acting *parallel to the base* of the cylinder) will be to *W* :: (*AB* : *AE*, upon the *plane*, and ∴ upon the *spiral line* ::) *EB* : *circumference AE* :: *distance between two threads of the spiral* : *circumference of the cylinder*.



3. By a spiral line thus raised upon the convex surface of a cylinder, is formed the *external* screw; and if upon the *concave* surface of a hollow cylinder of the same diameter a *groove* be cut to admit this protuberant spiral, it will form what is called the *internal* screw. Let the external and internal screw be fitted to each other in the

the manner represented in the annexed Figure (a), and let the *external* screw be turned round by a power applied to the lever *BC* (acting *parallel to the base* of the cylinder), whilst the *internal* screw remains fixed; then it is evident, from the manner in which the two screws act upon each other, that whilst the lever *BC* makes one revolution, the external screw will be *elevated* or *depressed* through one of the spaces *ab, bc, cd*, &c. according to the direction in which it is turned. When the screw is *depressed*,



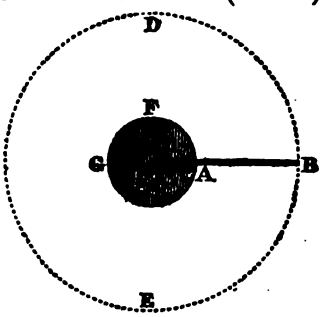
(a) In the machine here represented, the *internal* screw is contained in the perforation of the beam *AL*, which therefore must be considered as the part where the action of the screw takes place. It may be observed that the *form* of the spirals in this screw is not the same as that of the spirals upon the *cylinder* in the preceding Figure. The *flat* form there exhibited was only intended to explain the principle of the screw as immediately arising from that of the *inclined plane*. The principle of the screw does not depend so much upon the *form* as upon the *inclination* of the spirals. The most common form is that represented in the annexed Figure, where the spirals being made *tapering*, the action of the external and internal screw upon each other is rendered smooth and easy; but the *flat* form is most adapted to those cases where a *very great* pressure is to be created.

depressed, it drives before it the board EF, which moves in the grooves DE, NF, by which means a *pressure* is created upon any substance (S) placed between that board and the *fixt* board KM.

4. Let us now estimate the *quantity* of this pressure, by finding the relation which it bears to the power which produces it. To do this, it will be necessary, in the first place, to consider the force which would be generated in the *elevation* of the screw; which force may be estimated by shewing separately what part of it arises from the action of the *spirals of the screw* upon each other, and what from the action of the lever. As the machine turns round, *each point* of the external screw acts upon the *corresponding one* of the internal screw with a force analogous to that by which a body is sustained upon an inclined plane when the power acts *parallel to the base* of the plane; the *whole* force therefore of the screw will be of the same kind, and (by Art. 2.) will bear to the weight which it could support, the ratio of "*the distance between two spirals* : *circumference of the cylinder*." This would be the case, supposing the force to act *close to the surface of the cylinder*; when it acts therefore at the extremity of the lever BC, it will be increased in the proportion of the *radius of the lever* : *radius of the cylinder*.

5. Let d = *the distance between two spirals* of the screw, a = *length of the lever* (CB), p =

$p = 3.1415$ &c. then the circumference (BDE) of the circle described by the extremity of the lever $= 2pa$. Let P = the power acting at the *extremity of the lever*, p = the power acting at the *surface of the cylinder*, W = the weight which is kept in equilibrio by P, then, by Art. 4,



$p : W :: d$: circumference AFG[a],

$P : p :: CA : CB ::$ circumf. AFG : circumf. BDE;

$\therefore$ ex æquo, $P : W :: d$: circumf. BDE $(2pa)$,[b]

and $2paP = Wd$; $\therefore$ $P = \frac{Wd}{2pa}$, $W = \frac{2paP}{d}$,

$d = \frac{2paP}{W}$, and $a = \frac{Wd}{2Pp}$; hence, if any *three* of the four quantities W, P, a, d be given, the *fourth* may be found.

6. We

(a) AFG represents a section of the cylinder upon which the screw is cut.

(b) From this proportion it appears that the relation of $P : W$ depends entirely upon the distance between the spirals and the circumference which the power describes, *whatever be the thickness of the cylinder upon which the screw is cut*; and since $P \left(= \frac{Wd}{2pa}\right) \propto \frac{Wd}{a}$, when W is *given*, $P \propto \frac{d}{a}$; i.e. the power necessary to sustain a given weight varies *directly* as the *distance between the spirals*, and *inversely* as the *length of the lever*.

6. We have thus estimated the magnitude of the weight which might be sustained by a given power applied to the *elevation* of the screw; but this machine is more often used for the purpose of creating a *pressure*, than raising a *weight*; and it is evident, that whatever force is exerted by the screw to sustain a weight when it is turned in *one direction*, will also be exerted to create a *pressure downwards*, when it is turned in an *opposite* direction. Whether, therefore, the screw be applied to the *raising of a weight*, or the *creating a pressure*, the power necessary to sustain the *weight* or produce the *pressure* will always bear to that *weight* or *pressure* the ratio of "*the distance between any two spirals of the screw* "to *the circumference of the circle which the power* "*describes.*"

EXAM. I.

A screw, the distance between whose spirals is one inch, is turned horizontally by a lever whose length is 2 feet, reckoning from the axis of the screw; what weight could be sustained (or pressure produced) by it, when a power of 30 lbs. acts at the extremity of the lever?

By Art. 5, $W = \frac{2paP}{d} =$ (in the present instance, for $a = 24$ inch.) $\frac{2 \times 3.1415 \times 24 \times 30}{1} = 4523.76 lbs.$

i.e. a power of 30 *lbs.* applied to a machine of this kind would be sufficient to *sustain a weight*, or

or to *create a pressure*, equivalent to about 4523 *lbs*, or somewhat more than *two tons*.

EXAM. II.

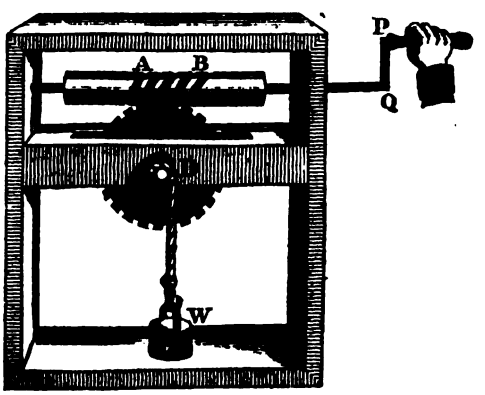
A person who could just lift a weight of 60 *lbs, found himself able, by means of a lever* 3 *feet long, acting as a handle to a screw, to sustain a ton weight; what was the distance between the spirals of the screw?*

By Art. 5, $d = \frac{2paP}{W} =$ (in the present case, for $a = 3$ feet) $\frac{2 \times 3.1415 \times 3 \times 60}{2240} = .504$ *feet* = about 6 *inches*.

EXAM. III.

In the annexed Figure, the screw (*AB*) *which is turned by a power* (*P*) *acting upon the handle* (*PQ*), *turns at the same time the wheel* (*C*), *in such a manner as to cause it to draw up the weight* (*W*) *by a rope wound round the axle* (*D*); *this is called the* endless *Screw, and it is required to find the ratio of P : W.*

Supposing all the parts of this machine to be nicely adjusted, it is a very powerful one; inasmuch as it combines the energy of the *Screw* with the multiplying power of the *wheel & axle*. To estimate its effect, therefore, let $PQ = a$, the distance between



between two spirals of the screw $= d$, radius of the *wheel* $= R$, radius of the *axle* $= r$, $p = 3.1415$, and $Q =$ the force exerted by the screw upon the wheel; then, by Art. 5,

$$P : Q :: d : 2pa,$$

and by Art. 2, page 255, $Q : W :: r : R$,

$$\therefore \text{ ex æquo, } P : W :: dr : 2paR.$$

Hence $P = \frac{Wdr}{2paR}$, and $W = \frac{2PpaR}{dr}$; let $d =$ one inch, $a = 12$ inches, $r = 4$ inches, $R = 18$ inches, and $P = 30$ *lbs.* then $W = \frac{40713.84}{4}$ *lbs.* $= 4.54$ tons; so that, by means of this machine, a power of 30 *lbs* would be sufficient to keep in equilibrio a weight of about $4\frac{1}{2}$ tons.

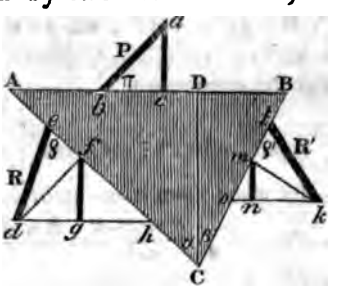
XXXVIII.

On the Wedge.

All those instruments which are used for the *separation of the parts of bodies*, such as *knives, axes, coulters, chissels*, &c. &c. come under the general denomination of the *wedge*; but these instruments are made of such variety of shapes, and forces are applied to them in such various manners, that, of all the mechanical powers, the *wedge* is that whose properties are least capable of being brought to a mathematical calculus. In the particular case when

when the wedge is of the form of a *triangular prism*, and the resistances upon its sides can be considered as *forces acting in given directions*, the relation between those resistances and the power which counteracts them may be estimated in the following manner.

1. Let *ABC* represent a section of the wedge perpendicular to the axis of the prism, and suppose its sides *AC, BC* to be perfectly *smooth*. Let a *power P*, whose magnitude and direction is represented by *ab*, act upon *AB* the *back* of the wedge, and let it be counteracted by two *resistances R, R'* (which are represented in quantity and direction by the lines *de, kl*) acting upon the *sides AC, BC*. Resolve *ab* into *ac perpendicular* and *bc parallel* to the back of the wedge, and let *de* also be resolved into *df* perpendicular and *ef parallel* to the side *AC*; and since the side *AC* is perfectly *smooth*, *df* only is effectual to stop the progress of the wedge; resolve *df* again into *dg parallel* and *gf perpendicular* to the *back* of the wedge, then *fg* is the only part of the resistance *R* which is directly opposed to that part of the power (viz. *ac*) which acts perpendicular to the back of the wedge. Let the resistance *R'* be resolved in the same manner, and let *mn* be that part of it which



is

is directly opposed to ac; then, supposing every part of the wedge to be perfectly *hard and inflexible*, it is evident that there will be an equilibrium between P and $R+R'$ when $ac=fg+mn$ [a].

2. Produce dg to h; and since, in the *right-angled* triangles, dfh, CAD, the *alternate* angles dhf, CAD are equal, the angle fdg must be equal to the angle ACD (α). Now

$de\ (R) : df :: \text{rad.} : \text{sin. } \angle def, \text{ or sin. } \rho,$
and $df : fg :: \text{rad.} : \text{sin. } \angle fdg, \text{ or sin. } \alpha,$

$$\therefore R : fg :: \overline{\text{rad.}}^2 : \text{sin. } \rho \times \text{sin. } \alpha, \text{ or } fg = \frac{R \times \text{sin. } \rho \times \text{sin. } \alpha}{\overline{\text{rad.}}^2}.$$

For the same reason $mn = \dfrac{R' \times \text{sin. } \rho' \times \text{sin. } \beta}{\overline{\text{rad.}}^2}$.

Again, $ab\ (P) : ac ::$ rad. : sin. π, $\therefore ac = \dfrac{P \times \text{sin. } \pi}{\text{rad.}}$.

But when there is an *equilibrium*, $ac=fg+mn$; Hence

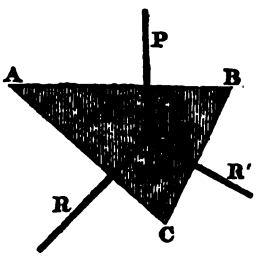
$$\frac{P \times \text{sin. } \pi}{\text{rad.}} = \frac{R \times \text{sin. } \rho \times \text{sin. } \alpha + R' \times \text{sin. } \rho' \times \text{sin. } \beta}{\overline{\text{rad.}}^2},$$

$$\text{or } P = \frac{R \times \text{sin. } \rho \times \text{sin. } \alpha + R' \times \text{sin. } \rho' \times \text{sin. } \beta}{\text{rad.} \times \text{sin. } \pi}.$$

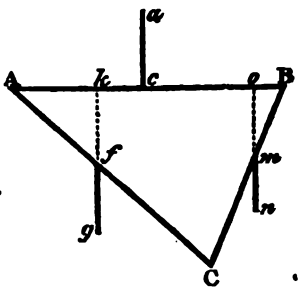
3. Suppose now the power (P) to act perpendicular to the *back*, and the resistances (R, R') perpendicular to the *sides* of the wedge; then sin. π, sin. ρ, sin. ρ' each become equal to *radius* ($=1$), and our equation becomes $P = R \times \text{sin. } \alpha + R' \times \text{sin. } \beta$. This

[a] But in ascertaining this relation between P and $R+R'$ when they are in equilibrio, it should be recollected that the point (c) cannot be *arbitrarily* assumed; for there evidently will be a tendency to *vibratory* or *rotatory* motion, unless the two resistances

This equation expresses the *general* relation between the power and the resistances when they are in *equilibrio*, and act *perpendicularly* upon the back and sides of a wedge. But in the *particular* case, when the directions of P, R, R' meet in a *point*, this relation may be expressed more simply; for if those directions meet in D, then it is evident that this equilibrium is produced under the same circumstances as that "of a body kept *at rest* by three forces whose directions meet in that point;" but (by Art. 3, page 71) those three forces would be to each other as the three sides of a triangle *perpendicular* to their respective directions; P, R, and R' will therefore be to each other *as the three sides of*



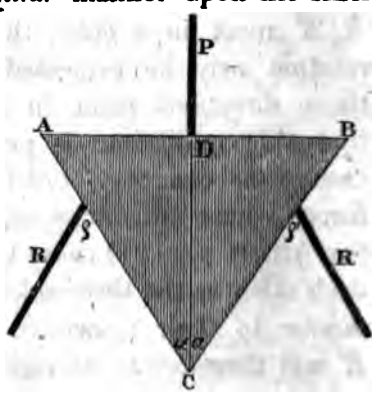
resistances *balance themselves* about that point. To determine the point c so that this tendency to vibratory motion shall be *prevented*, produce gf, nm to k and o, then it is evident that the resistances fg, mn will only balance themselves about c (which may in this case be considered as the *center of motion* of an inverted lever which is supported by a power or *prop* acting in direction ac) when $fg \times ck = mn \times co$, or $ck : co :: mn : fg$. That no vibratory or rotatory motion may be produced, it appears therefore that P must act upon the back of the wedge in such a manner, that the point at which it acts may cut ko in the ratio of $mn : fg$.



of the triangle ABC; i.e. $P : R :: AB : AC$; $P : R' :: AB : BC$; and, consequently, $P : R + R' :: AB : AC + BC$:: *back of the wedge* : *sum of the sides*[a].

4. If the wedge be *isosceles*, and two *equal* resistances, acting in a *similar* manner upon the sides AC, BC, are kept in equilibrio by a power acting *perpendicular* to the back AB, then sin. α = sin. β; $R' = R$; sin. ρ' = sin. ρ; and sin. π = rad.; our equation therefore becomes

$$P = \frac{2R \times \text{sin.}\rho \times \text{sin.}\alpha}{\overline{\text{rad.}}^2};$$



hence $P : 2R$:: sin. $\rho \times$ sin. $\alpha : \overline{\text{rad.}}^2$; i.e. "the "*power* : *sum of the resistances* :: *sine of the* ∠ "*which*

(a) Let $\angle A = \alpha$, $\angle B = \beta$, then $\angle C = \overline{180^\circ - \alpha + \beta}$ (Fig. in Art. 3, page 293); $\therefore AB : AC$:: sin. $\overline{180^\circ - \alpha + \beta}$: sin. β, and $AB : BC$:: sin. $\overline{180^\circ - \alpha + \beta}$: sin. α; hence $AB : AB + BC$:: sin. $\overline{180^\circ - \alpha + \beta}$: sin. α + sin. β; from which it appears that when the power and resistances act *perpendicularly* to the back and sides of a wedge, there is an analogy between the case of equilibrium of P, R, R', and that of a body supported between *two inclined planes*; for the sides of the triangle ABC correspond to the sides of the triangle MCK in Cor. 1. Exam. 5. page 282. This is the analogy alluded to in Note (a), page 228.

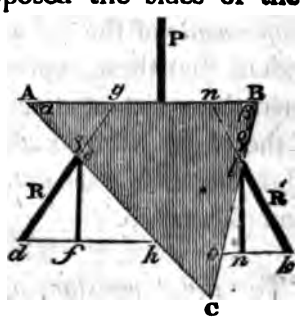
"*which the resistances make with the side of the* "*wedge* × *sine of* $\frac{1}{2}$ *the vertical* $\angle$ *of the wedge* " : *square of the radius.*" Moreover,

I. Let the resistances be *perpendicular to the sides*, then sin. ρ = rad. $\therefore P : 2R ::$ sin. α : rad. $:: AD : AC$.

II. If the resistances act *perpendicular to the back*, then R becomes parallel to CD, $\therefore \rho = \alpha$; $P : 2R ::$ sin. α^2 : rad.$^2 :: AD^2 : AC^2$.

III. Let the resistances act *parallel to the back*, then $180^0 - \rho = CAD$ = complement of ACD, $\therefore$ sin. $180^0 - \rho$ or sin. ρ = cos. α; hence $P : 2R$:: sin. $\alpha \times$ cos. α : rad.^{2} :: $\frac{1}{2}$ rad. × sin. 2α[b] : rad.$^2 :: \frac{1}{2}$ sin. 2α : rad. $:: \frac{1}{2}$ sin. $\angle ACB$: *radius.*

5. Hitherto we have supposed the sides of the wedge to be perfectly *smooth*; but conceive them to be of such a nature that the resistances are *wholly effective*,[c] then the relation between P and $R + R'$ must be estimated in a different manner; for in this case it will usually be necessary to resolve *de* into *df* parallel



(b) For rad. : cos. α :: 2 sin. α : sin. 2α, $\therefore$ sin. $\alpha \times$ cos. α = $\frac{1}{2}$ rad. × sin. 2α. *Trigonometry*, Prop. 3. page 9.

(c) i. e.; if (on account of the *roughness* of its sides, or from *any other cause*) the wedge has no tendency to *slide* in the directions AC, BC.

parallel and *ef perpendicular* to the back of the wedge, and *ef* will be the part of the *resistance* directly opposed to the *power* acting at *right angles* to the back; let *kl* be resolved in the same manner, and *ln* will be the part of the resistance *R'* which is opposed to *P*; hence, when there is an equilibrium, $P = ef + ln$. Now $de\ (R) : ef :: \text{rad. } (1) : \text{sin. } \angle edf$ or $\angle Age$:: (for $\angle Age = 180^\circ - \overline{\angle Aeg + e\angle Ag}$) $1 : \text{sin. } 180^\circ - \overline{\alpha + \rho}$, $\therefore\ ef = R \times \text{sin. } 180^\circ - \overline{\alpha + \rho}$; for the same reason $ln = R' \times \text{sin. } 180^\circ - \overline{\beta + \rho'}$; $\therefore\ P = R \times \text{sin. } 180^\circ - \overline{\alpha + \rho} + R' \times \text{sin. } 180^\circ - \overline{\beta + \rho'}$. If the resistances act *perpendicular to the sides*, then ρ, ρ' are each equal to 90°, $\therefore\ P = R \times \text{sin. } \overline{90^\circ - \alpha} + R' \times \text{sin. } \overline{90 - \beta} = R \times \text{cos. } \alpha + R' \times \text{cos. } \beta$; which coincides with the expression in Art. 3, for α, β in the present case are the *complements* of the $\angle^s\ \alpha, \beta$ in that article; and it is evident that these expressions *ought* to coincide, for when the resistances act *perpendicular* to the sides of the wedge, it can make no difference whether the sides be *smooth* or *not*. We now proceed to an example or two.

EXAM. I.

Two equal resistances (R, R') *acting at angles of* 60° *and* 30° *upon the sides of a perfectly smooth wedge, are kept in equilibrio by a power acting perpendicular to its back; the angle which a perpendicular (from the vertical angle of the wedge to the back) makes with the side upon which R acts is* 45°, *and with the side upon which R' acts* 30°; *required the ratio of* $P : R + R'$ *or* $2R$.

This

This is a case of Art. 2, where $R' = R$, sin. π = rad. (which let) $= 1$, $\therefore P = R.\overline{\sin.\rho \times \sin.\alpha + \sin.\rho' \times \sin.\beta}$; But,

$$\left.\begin{array}{l}\sin.\rho = \sin.60^\circ = \frac{\sqrt{3}}{2}\\ \sin.\alpha = \sin.45^\circ = \frac{1}{\sqrt{2}}\\ \sin.\rho' = \sin.30^\circ = \frac{1}{2}\\ \sin.\beta = \sin.30^\circ = \frac{1}{2}\end{array}\right\} \therefore P = R.\overline{\frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{4}},$$

$$= \frac{R.\overline{\sqrt{6}+1}}{4}; \text{ hence}$$

$$P : R :: \sqrt{6}+1 : 4, \text{ \& } P : 2R :: \sqrt{6}+1 : 8.$$

EXAM. II.

A power, acting perpendicularly upon the back of a perfectly smooth isosceles wedge, is kept in equilibrio by two equal resistances acting at angles of 60° *upon the sides of the wedge; the vertical angle of the wedge is a right angle; what is the ratio of P to the sum of the resistances?*

By Art. 4, $P : 2R ::$ sin. $\rho \times$ sin. $\alpha :$ $\overline{\text{rad.}}^2$: in the present instance sin. ρ = sin. 60° $= \frac{\sqrt{3}}{2}$, sin. α = sin. 45° $= \frac{1}{\sqrt{2}}$, rad. $= 1$, $\therefore P : 2R :: \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} \cdot 1 :: \sqrt{3} : 2\sqrt{2}$.

XXXIX.

In estimating the relation between the power and the weight in this and the two preceding Lectures,

[illegible] we have only considered the case when they are in a state of *equilibrium*. To put the machine in motion, the *power* must evidently be increased[*]: in which case the *velocity of the power* will bear to the *velocity of the weight* the same ratio which the *weight* bears to the *power* when they are in *equilibrio*; the velocity of the power and weight being estimated in the direction in which they respectively act.

1. In the *straight lever* (see Fig. page 348), since the arcs Pp, Ww are described in the *same time* by P and W respectively, the *velocity of* P : *velocity of* W :: $Pp : Ww :: Aa : Bb :: CA : CB :: W : P$ when (P) and (W) are in *equilibrio*.

2. In *the wheel and axle*, whilst the power descends through a space equal to the *circumference of the wheel*, the weight ascends through a space equal to the *circumference of the axle*, ∴ the *velocity of the power* : the *velocity of the weight* :: the circumference

(*) In estimating the increase of the power necessary to put the machine in motion, the *friction* of its different parts should be taken into the account. If p=the sum of the impediments arising from friction, and P=the power which would keep the weight (W) in *equilibrio*, then it is evident, that, before the machine can be put in motion, the power actually employed must exceed $P+p$.

ference of the *wheel* : the circumference of the *axle* :: *radius of the wheel* : *radius of the axle* :: W : P.

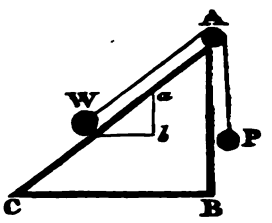
3. In the *single fixt pulley*, the power and weight move through *equal* spaces in the *same time*, $\therefore$ the velocity of the *power* is equal to the velocity of the *weight*; and in this machine $P = W$, $\therefore$ velocity of power : velocity of weight :: W : P.

4. In the system of n pulleys (Fig. page 263), whilst the power descends through any space (x), *each* of the strings belonging to the block of pulleys to which the weight is appended is shortened by $\frac{1}{n}x$, $\therefore$ the weight ascends through a space equal to $\frac{1}{n}x$ in the same time that (P) descends through the space x; hence the velocity of the *power* : the velocity of the *weight* :: $x : \frac{1}{n}x :: n : 1 :: W : P$.

5. In the system of pulleys (Fig. page 264), whilst P descends through any space (x), the *pulley* F is raised through a space $= \frac{1}{2}x$; the *pulley* D through a space $= \frac{1}{2} \times \frac{1}{2}x = \frac{1}{4}x$; the *pulley* B through a space $= \frac{1}{2} \times \frac{1}{4}x = \frac{1}{8}x$; &c. &c. hence (if n = the number of pulleys) velocity of P : velocity of $W :: x : \frac{1}{2^n} \times x :: 2^n : 1 :: W : P$.

6. Let ACB be an inclined plane, up which the weight (W) is drawn by a power (P) acting over a pulley

a pulley at A, then $P : W :: AB : AC$. Draw Wb parallel to CB, and ab parallel to AB, then whilst P descends through a space equal to Wa, W ascends upon the *plane* through the *same space*; but the space



actually described by (W) in this time in the *direction of gravity*, is ba; $\therefore$ (when the velocity of the power and weight are estimated in the direction in which they *respectively act*) the velocity of P : the velocity of $W :: Wa : ab ::$ (by sim. Δ's) $AC : AB :: W : P$.

7. In the *screw*, whilst P describes the circumference of a circle whose radius is BC (Fig. in page 287), the *weight* is elevated or depressed through a space equal to *the distance between two contiguous spirals* (d), $\therefore$ the velocity of P : the velocity of $W ::$ circumference $(BDE) : d :: W : P$.

In the same manner we might trace this relation between the *velocity of the power* and the *velocity of the weight* in machines *where the power and weight act obliquely to each other*; but as the operation then becomes somewhat more intricate, and as what has been already shewn is sufficient to establish the truth of our proposition, it seems unnecessary to pursue the investigation any further.

Lecture X.

ON THE

PRESSURE AND TENSION OF CORDS.

THE pressure and tension of Cords, arising from the action of forces employed in Machinery, may be considered in two points of view; I. *When the cords pass freely over wheels or pulleys.* II. *When they are fixt at both ends*(a). Each of these cases will require a distinct investigation.

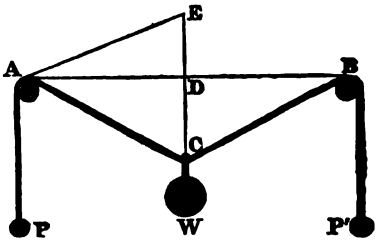
XL.

On the Pressure and Tension of Cords passing freely over Pulleys.

1. Let *A* and *B* be two pulleys fixt at a given distance from each other in the same horizontal line,

(a) When the cord is fixt at *one* end, and (passing over a pulley or pulleys) is acted upon by a power or weight at the *other*, it constitutes a *third* case; but this case evidently belongs to that of a "*pulley or system of pulleys*," which has already been considered in our 8th Lecture.

line, and suppose a cord $PAWBP'$ to pass over them, at the extremities of which are suspended the two *equal* weights P, P'; these two weights being kept in equilibrio by a *third* weight W[a]. Draw AE parallel to CB, and produce WC to meet it in E, then the three sides CA, AE, EC of the triangle ACE will represent the quantity and direction of the three forces by which the weight W is kept *at rest*; for these three forces are, 1st, The *tension produced by* P in direction CA; 2dly, The *tension produced by* P in direction CB or AE; and, 3dly, its *own gravity* in direction EC; and since $P' = P$, AE or CB must be equal to AC. Hence P or $P' : W :: AC$ or $CB : EC$.



2. Join AB; then, since AB is *parallel* and CE *perpendicular* to the horizon, the angles at D are *right* angles, and the lines AB, CE *bisect* each other in D[b]. Hence $EC = 2CD$, and $P : W :: AC : (EC$ or$)\ 2CD$; $\therefore 2P : W :: 2AC :: 2CD :: AC : CD$, and $4P^2 : W^2 :: AC^2 : CD^2$; also,

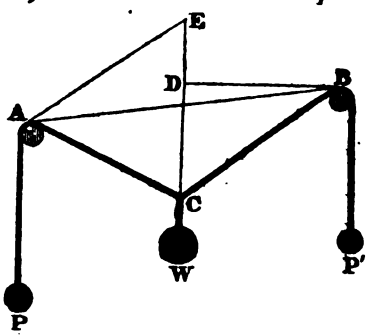
$4P^2$

(a) In this and the following articles we suppose the cords to be *without weight*, and perfectly *inextensible*; and the pulleys to be so *small*, and so adjusted, that the centers of gravity of P, P', W &c. may all lie in the *same vertical plane*.

(b) ACE and ACB being *isosceles* triangles.

$4P^2 - W^2 : W^2 :: AC^2 - CD^2 (AD^2) : CD^2$, $\therefore CD^2 = \frac{W^2 \times AD^2}{4P^2 - W^2}$, or $CD = \frac{W \times AD}{\sqrt{4P^2 - W^2}}$; which gives the position of W when the equilibrium takes place, for P, W and AD ($= \frac{1}{2}AB$) are *known* quantities[c].

3. Suppose now that AB is not *parallel to the horizon* (but makes some *given* angle with the horizontal line BD), and that P' is not *equal* to P; then draw AE parallel to CB, and produce WC to meet it in E. For the same reason as before, the three sides CA, AE, EC of the triangle AEC will represent the quantity and direction of the three forces P, P', W, by which the equilibrium is effected; and since the *proportion* of the three sides of this triangle is known, its *three angles* may be found. Let $CAE = \alpha$, AEC or $ECB = \beta$; then since AE is parallel to CB, the $\angle ACB = 180^\circ - CAE = 180^\circ - \alpha$. Now the $\angle DBC = 90^\circ - ECB = 90^\circ - \beta$; if therefore

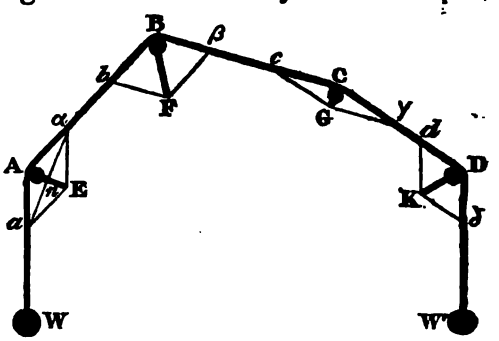


$\angle DBA$

(c) If $P = \frac{1}{2}W$, or $2P = W$, then $4P^2 = W^2$, and $4P^2 - W^2 = 0$; if W be *greater* than $2P$, then $4P^2 - W^2$ is *negative*; in the former case, therefore, the value of CD becomes *infinite*, and in the latter *impossible*; which shews that if W be *equal to* or *greater than* $2P$, then *no equilibrium* can take place.

$\angle DBA=\gamma$, we have $\angle ABC=(DBC-DBA=)$ $90-\beta-\gamma$; hence, in the triangle ABC, the side AB and the two angles ACB, ABC are known, from which the other sides AC, CB (and consequently the position of C when there is an *equilibrium*) may be determined [a].

4. Let us next take the case of two *equal* [b] weights W,W' balancing themselves over any number of pulleys $A, B, C,$ &c. and let it be required to find *the pressure upon each pulley*.



Since the tension of the cord $WABCDW'$ is everywhere the *same*, take Aa, $A\alpha$, Bb, $B\beta$, &c. equal to one another on each side of

[a] This Problem (like the *former*) is in course limited to a *certain relation* between W and $P+P'$. This limitation must be discovered in the solution of each particular case; for it is evident that if the angles of the triangle AEC should be such as to give the angle ABC *equal to or greater than the given angle* ABP', the line BC will *never meet* the vertical line EW, and therefore the case becomes *impossible*.

[b] It is evident that this Problem is limited to the case of two *equal* weights; for since the cord passes *freely* over all the pulleys, if the weights be *unequal* then no equilibrium could ensue.

of the pulleys A, B, C, &c. and complete the parallelograms $AaE\alpha$, $BbF\beta$, &c; draw the *diagonals* AE, BF, CG, &c. and (since the forces acting on the pulleys are represented by the two adjoining sides of each parallelogram) these diagonals will represent the *pressure* upon the several pulleys A, B, C, &c. Now the parallelograms $AaE\alpha$, $BbF\beta$, &c. having all their sides *equal*, their diagonals $AE, a\alpha$; $BF, b\beta$; &c. will cut each other at *right angles*, and likewise *bisect* each other; hence AE, BF, CG, &c. are equal to *twice the cosine of half the angles* $aA\alpha$, $bB\beta$, &c. (to radius Aa or Bb, &c.) respectively; if therefore those angles be denoted by π, π', π'', &c. and radius = 1, the pressures upon the pulleys A, B, C, &c. will be denoted by $2W \times \cos. \frac{1}{2}\pi$, $2W \times \cos. \frac{1}{2}\pi'$, $2W \times \cos. \frac{1}{2}\pi''$, &c.[c]

EXAM. I.

In the Figure, page 302, *let* $AB = d$, *and* $P : W :: 1 : n$ (*n being less than* 2); *it is required to find the position of* W, *when there is an equilibrium between* W *and* $P + P'$.

Since

(c) Take, for instance, the parallelogram about the pulley A; then $AE = 2An = 2 \sin. \angle Aan = 2 \cos. \angle aAn = 2 \cos. \frac{1}{2} aA\alpha = 2 \cos. \frac{1}{2}\pi$; but W : pressure on pulley A :: Aa : AE or $2An$:: rad. (1) : $2 \cos. \frac{1}{2}\pi$, $\therefore$ pressure on pulley $A = 2W \times \cos. \frac{1}{2}\pi$; and so of the rest.

If there be *three* pulleys A, B, D, placed in the form of an *isosceles* triangle whose base AD is parallel to the horizon and $\angle ABD = 120^\circ$, then each of the angles at A and D is *also* equal to

Since $AB=d$, $AD=\frac{1}{2}d$; and since $P:W::1:n$, $P=\frac{W}{n}$, $\therefore\ 4P^2=\frac{4W^2}{n^2}$, and $4P^2-W^2=\frac{4W^2}{n^2}-W^2=\frac{W^2.\overline{4-n^2}}{n^2}$; hence $CD^2\left(=\frac{W^2\times AD^2}{4P^2-W^2}\text{ by Art. 2.}\right)=\frac{n^2W^2\times\frac{1}{4}d^2}{W^2.\overline{4-n^2}}=\frac{n^2\times d^2}{4.\overline{4-n^2}}$, and $CD=\frac{nd}{2\sqrt{4-n^2}}$; bisect therefore AB in D, and take $CD=\frac{nd}{2\sqrt{4-n^2}}$, and it will give the position of W when the equilibrium takes place.

Cor. Let $P=W$, then $n=1$, and $CD=\frac{AB}{2\sqrt{3}}$; let $P:W::1:\sqrt{2}$, then $n=\sqrt{2}$, and $CD=\frac{AB}{2}$; &c. &c.

EXAM. II.

In the Figure of Art. 3, page 303, *the weights* P, P', *and* W *are respectively* 3, 4, *and* 5 *lbs; the distance* (AB) *between the pulleys is* 6 *feet, and it is inclined to the horizontal line* BD *in an angle* (DBA) *of* 30 *degrees; it is required to find the position of* W *when the equilibrium takes place?*

Since

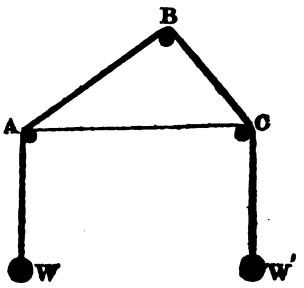
to 120°; hence $\cos.\frac{1}{3}\pi$ ($=\cos.\frac{1}{3}\pi'=\cos.\frac{1}{3}\pi''$) $=\cos.60°=\sin.30°=\frac{1}{2}$, and $2W\times\cos.\frac{1}{3}\pi=2W\times\frac{1}{2}=W$; in this case the pressure upon each of the three pulleys is the *same*, and just equal to either of the weights W or W'.

Since the three sides CA, AE, EC of the triangle AEC are as (P, P', W, i.e. as) the numbers 3, 4, 5 respectively, the angle CAE will be 90°, and the angle AEC 36°.54′; hence the angle ACB is 90°, and the angle ABC ($=90°-\beta-\gamma=$) $90°-36°.54'-30°=23°.6'$; from which the two sides AC, CB of the triangle ABC (AB being 6 feet) are found to be 2.354 and 5.518 feet respectively (a).

EXAM. III.

Two equal weights W, W' balance each other over the three pulleys A, B, C, placed in the form of a triangle whose base AC is parallel to the horizon; and the angles ABC, BCA, BAC, are 90°, 60°, 30° respectively; it is required to compare the pressures upon A, B, C.

Since $BAC=30°$, and $BCA=60°$, the three angles WAB, ABC, BCW' are respectively 120°, 90° and 150°; hence, by Art. 4, the pressure upon $A=2W\times$ cos.60°, upon $B=2W\times$ cos.45°, and upon $C=2W\times$ cos.75°, ∴ the pressures upon A, B, C are as cos. 60°,



cos.

(a) If in this and the *preceding* example it had been required to find the *pressure* upon the pulleys A and B, this might easily be

cos. 45°, and cos. 75°, or as sin. 30°, sin. 45°, sin. 15°; but if rad. = 1, sin. $30° = \frac{1}{2}$, sin. $45° = \frac{1}{\sqrt{2}}$, and sin. $15° = \frac{\sqrt{2-\sqrt{3}}}{2}$ [a], ∴ (multiplying these numbers by 2) it appears that the pressures upon A, B, C are as the numbers 1, $\sqrt{2}$, and $\sqrt{2-\sqrt{3}}$ respectively.

Cor. If the *absolute* pressure upon each pulley was required, then these numbers must be multiplied by $2W$; thus if $W = 10$ *lbs*, then the *pressure upon* $A = 20 \times \frac{1}{2} = 10$ *lbs*; *pressure upon* $B = 20 \times \frac{1}{\sqrt{2}} = 14.14$ *lbs*; *pressure upon* $C = 20 \times \frac{\sqrt{2-\sqrt{3}}}{2} = 5.17$ *lbs*.

XLI.

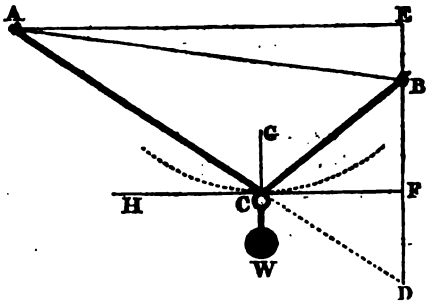
On the Pressure and Tension of Cords, fastened at both ends.

1. Suppose the cord ACB to be fastened to the hooks A and B, and let a weight (W), suspended

be effected by means of the expressions in Art. 4, page 304; for when the point C is determined, the angles PAC, PBC will be known, and the *pressure upon pulley* $A = 2P \times \cos. \frac{1}{2} PAC$, the *pressure upon* $B = 2P \times \cos. \frac{1}{2} PBC$.

[a] For $\sin \frac{1}{2}\alpha = \sqrt{\frac{1-\cos. \alpha}{2}}$, ∴ $\sin. 15° = \sqrt{\frac{1-\cos. 30°}{2}} = \sqrt{\frac{1-\sqrt{\frac{1}{2}}}{2}} = \sqrt{\frac{2-\sqrt{3}}{2}}$. See *Plane Trigonometry*, page 16.

pended by the *ring C*, slide freely along it; it is evident that whilst the weight descends, this ring will describe an *ellipse* (whose *foci* are *A* and *B*), to which the line *WC* will be a *normal* at the *lowest* point *C*. Produce *WC* to *G*, and through *B* draw *ED* parallel to *GW*; draw the tangent *HF* (which will evidently be a *horizontal* line), and *AE* parallel to it; and produce *AC* to meet *ED* in *D*. By the property of the ellipse, *CG bisects* the angle *ACB*, ∴ the angle *CBD* (= *GCB*) is equal to the angle *CDB* (= *ACG*); hence *CD* = *CB*, and consequently *AD* = (*AC* + *CD* =) *AC* + *CB* = the *length of the cord*. To determine the *lowest point C* therefore, we have only to draw *BD* perpendicular to the horizon, and with *center A* and *radius* = length of the cord to describe a circular arc cutting *BD* in *D*; then *bisect BD*[c] in *F*, and draw *FC* at right angles to *BD*, and the point *C* where it cuts *AD* will be the point required.



2. It is evident that the weight will *rest* when it arrives at the lowest point, and that the three forces which *keep* it at rest will be represented by the three sides

(c) Since *CBD* is an *isosceles* triangle, the tangent *HF*, which is *perpendicular* to *BD*, also *bisects* it.

sides DC, CB, BD of the triangle CBD; i.e. the *pressure or tension upon* A will be represented by DC, the *pressure or tension upon* B by CB. and the *weight itself* by B; and since $CB=CD$, these pressures will be *equal*; hence the *pressure upon* A *or* B : W :: CD or CB : BD :: CD : $2DF$. For the purpose of estimating this pressure, let (AD) the *length of the string* $=l$, the *given line* AB $=a$, and the angle BAE which it makes with the *horizontal* line $AE=\alpha$; then $AE=a\times\cos.\alpha$, and $DE=\sqrt{AD^2-AE^2}=\sqrt{l^2-a^2\times\cos.\alpha^2}$; but by similar triangles CD : DF :: AD : DE, $\therefore$ CD : $2DF$:: AD : $2DE$; hence *pressure on* A *or* B (Pr.) : W (:: CD : $2DF$) :: AD : $2DE$:: l : $2\sqrt{l^2-a^2\times\cos.\alpha^2}$, and $\text{Pr.}=\frac{Wl}{2\sqrt{l^2-a^2\times\cos.\alpha^2}}$ (*).

3. Let us next suppose the cord ACB to be kept stretched by a *power* (P) acting upon it in some *given* direction PCG, the cord still passing freely through

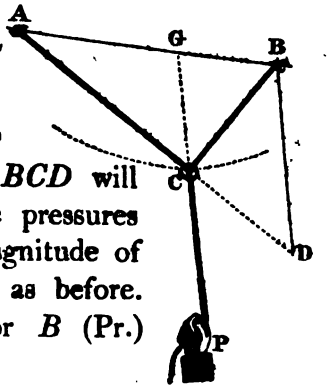
(*) If $\alpha=0$, or the line AB be *horizontal*; then cos. α=rad. $=1$, and $P=\frac{Wl}{2\sqrt{l^2-a^2}}$; and as in this case $DF=\frac{1}{2}DE$, DC will be equal to $\frac{1}{2}AD$, $\therefore$ $AC=CD=CB$. If $\alpha=0$, and $l=a$, then, whatever be the value of W, P will be *infinite*; i.e, "if a weight "however *small* be suspended from a cord fastened to two hooks "whose horizontal distance is equal to the *length* of the cord, "then it will require an infinite force from those hooks to keep "the cord *perfectly straight*." This difficulty of keeping a cord or chain perfectly straight, even if acted upon only by its *own* weight (supposing it to *have* weight), is a matter of common experience.

sides DC, CB, BD of the triangle CBD; i.e. the *pressure or tension upon* A will be represented by DC, the *pressure or tension upon* B by CB, and the *weight itself* by B; and since $CB = CD$, these pressures will be *equal*; hence the *pressure upon* A *or* $B : W :: CD$ or $CB : BD :: CD : 2DF$. For the purpose of estimating this pressure, let (AD) the *length of the string* $= l$, the *given line* $AB = a$, and the angle BAE which it makes with the *horizontal* line $AE = \alpha$; then $AE = a \times \cos.\alpha$, and $DE = \sqrt{AD^2 - AE^2} = \sqrt{l^2 - a^2 \times \overline{\cos.\alpha}^2}$; but by similar triangles $CD : DF :: AD : DE$, $\therefore CD : 2DF :: AD : 2DE$; hence *pressure on* A *or* B (Pr.) $: W$ $(:: CD : 2DF) :: AD : 2DE :: l : 2\sqrt{l^2 - a^2 \times \overline{\cos.\alpha}^2}$, and $\text{Pr.} = \dfrac{Wl}{2\sqrt{l^2 - a^2 \times \overline{\cos.\alpha}^2}}$ (a).

3. Let us next suppose the cord ACB to be kept stretched by a *power* (P) acting upon it in some *given* direction PCG, the cord still passing freely through

(a) If $\alpha = 0$, or the line AB be *horizontal*; then cos. α = rad. $= 1$, and $P = \dfrac{Wl}{2\sqrt{l^2 - a^2}}$; and as in this case $DF = \frac{1}{2} DE$, DC will be equal to $\frac{1}{2} AD$, $\therefore AC = CD = CB$. If $\alpha = 0$, and $l = a$, then, whatever be the value of W, P will be *infinite*; i.e, "if a weight "however *small* be suspended from a cord fastened to two hooks "whose horizontal distance is equal to the *length* of the cord, "then it will require an infinite force from those hooks to keep "the cord *perfectly straight*." This difficulty of keeping a cord or chain perfectly straight, even if acted upon only by its *own* weight (supposing it to *have* weight), is a matter of common experience.

through the *ring* C. In this case, also, the line PCG will be a *normal* to the ellipse in the point C; and if BD be drawn parallel to GP, and AC produced to meet it in D, the three sides DC, CB, BD of the *isosceles* triangle BCD will respectively represent the pressures upon A or B and the magnitude of the weight or power P, as before. Hence pressure on A or B (Pr.) : P :: CB or CD : BD.

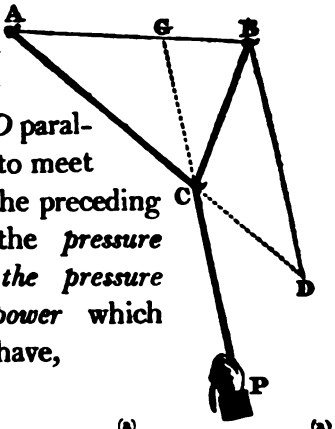


4. To estimate this pressure, let $AD = l$, $AB = a$, and $\angle PGB = \alpha$; then $AD(l)$: $AB(a)$:: sin. ABD or sin. PGB (sin. α) : sin. $\angle ADB = \frac{a \times \text{sin. } \alpha}{l}$; the $\angle ADB$ therefore is known; let this angle $= \beta$, then $\angle ACB = 2\beta$; but Pr. : P :: CB : BD :: sin. CDB : sin. BCD or sin. ACB :: sin. β : sin. 2β, $\therefore$ Pr. $= \frac{P \times \text{sin. } \beta}{\text{sin. } 2\beta} = \frac{P \times \text{sin. } \beta}{2 \text{ sin. } \beta \times \text{cos. } \beta} = \frac{P}{2 \text{ cos. } \beta}$ $= \frac{P}{2 \text{ cos. } \frac{1}{2} ACB}$.

5. But the case is quite different, if, instead of a *ring*, the force acts by means of a *knot* in the cord. For instance, let the string PC be fastened in a *knot* at C, and let a force (P) act in some given direction PCG to keep the cord in the position

 ACB.

ACB. In this case it is evident that the line *PG* does not necessarily *bisect* the angle *ACB*. Draw *BD* parallel to *PG*, and produce *AC* to meet it in *D*, then since (as in the preceding Article) *DC* represents the *pressure upon the hook A*, *BC* *the pressure upon B*, and *BD* the *power* which produces this pressure, we have,



Pr. on $A : P :: DC : BD ::$ sin. CBD or GCB or BCP[a] : sin. BCD or ACB[a],

and

$P :$ Pr. on $B :: BD : BC ::$ sin. BCD or ACB : sin. BDC or ACG or ACP[a],

$\therefore$ ex æquo,

Pressure on A : Pressure upon B :: sin. BCP : sin. ACP;

i.e, "the pressures upon the hooks A and B are "to each other *inversely* as the sines of the angles "which the direction of the power (PC) makes "with that part of the cord to which they are re-"spectively attached."[b]

6. Since

(a) BCP, ACB and ACP being respectively the *supplements* of GCB, BCD and ACG.

(b) We have here supposed the cord to be kept stretched by means of a power P, but the same would evidently be true of a *weight* (W) suspended by a *knot* in the cord; so that (in Fig. page 309) the pressures on A and B would not necessarily be *equal*, but *pressure on A* would be to *pressure on B* :: sin. BCW : ACW; also pressure on A : W :: sin. BCW : ACB, and pressure on B : W :: sin. ACW : sin. ACB.

6. Since the knot is tied at a *given* point in the cord, and the hooks are placed at a *given* distance from each other, the three *sides* AC, CB, AB of the triangle ACB are known, and consequently the three *angles* may be found. Hence the angle ACB is known; and since CGB is known[c], the $\angle BCP$ $(= CGB + CBG)$ will also be known; from which $ACP (= 360^\circ - \overline{ACB + BCP})$ may be found. Having thus obtained the angles ACP, ACB, BCP, we are enabled to compare the pressures upon A and B with each other, and with the power (P).

EXAM. I.

Let a weight of 10 *lbs, sliding freely along a cord whose length is* 8 *feet, be supported by two hooks placed at the distance of* 6 *feet from each other, and in a line making an angle of* 30° *with the horizon; required the pressure upon each hook.*

This is a case of Art. 2, page 310, where $W = 10 lbs$, $l = 8$ feet, $a = 6$ feet, $\alpha = 30^\circ$ (and $\therefore$ cos. α = sin. $60^\circ = \frac{\sqrt{3}}{2}$); hence the *pressure upon each hook* $\left(= \frac{Wl}{2\sqrt{l^2 - a^2 \times \cos.\alpha^2}} \right) = \frac{10 \times 8}{2\sqrt{64 - 27}}$

$= \frac{80}{12.10} = 6.57 lbs.$

Ex.

(c) For PCG is supposed to be inclined to AB in a *given* angle.

EXAM. II.

A cord of 12 *feet in length is tied to two hooks whose distance from each other is* 8 *feet; this cord passes through a ring upon which a power of* 50 *lbs. acts in a direction making an* $\angle$ *of* 60° *with a line drawn from one hook to the other; what is the pressure upon each hook?*

By referring to Art. 4, (Figure in page 311) we have $AB(a) = 8$ feet, $AD(l) = 12$ feet, and $\alpha = 60°$, $\therefore$ sin. $ADB \left(= \frac{a \times \text{sin. } \alpha}{l} \right) = \frac{8 \times \text{sin. } 60°}{12} = \text{sin. } 35°\ 16' = \beta$; hence the *pressure upon each hook* $= \left(\frac{P}{2 \cos. \beta} = \right) \frac{50}{2 \cos. 35°.16'} = \frac{50}{1.6328} = 30.62$ *lbs.*

EXAM. III.

A cord ACB, *whose length is* 14 *feet, is fastened to two hooks* A *and* B *at the distance of* 12 *feet from each other (see Figure in Art.* 5, *page* 312); $AC = 9$ *feet,* $BC = 5$ *feet, and another cord* PC *is fastened to* ACB *by a knot at* C; *it is then stretched by a power* (P) *of* 60 *lbs, acting in direction* PG, *making an angle of* 70° *with* AB; *compare the pressures upon the hooks with each other and with the power?*

Since $AB = 12$, $AC = 9$, $CB = 5$, the angle ACB will be equal to 114°.57′, and ABC to 42°.51′; hence, since $CGB = 70°$, BCP ($= ABC$

+

$+ CGB) = 42°.51' + 70° = 112°.51'$; from which we have $ACP\ (= 360° - \overline{ACB + BCP}) = 360° - \overline{114°.57' + 112°.51'} = 132°.12'$. But, by Art. 5,

Pressure on A : P (60) :: sin. BCP : sin. ACB
:: sin. $112°.51'$: sin. $114°.57'$
:: .9215 : .9066,
$\therefore$ pressure on $A = 60.9$ *lbs.*

Again,

Pressure on B : P (60) :: sin. ACP : sin. ACB
:: sin. $132°.12'$: sin. $114°.57'$
:: .7408 : .9066,
$\therefore$ pressure on $B = 49.02$ *lbs.*

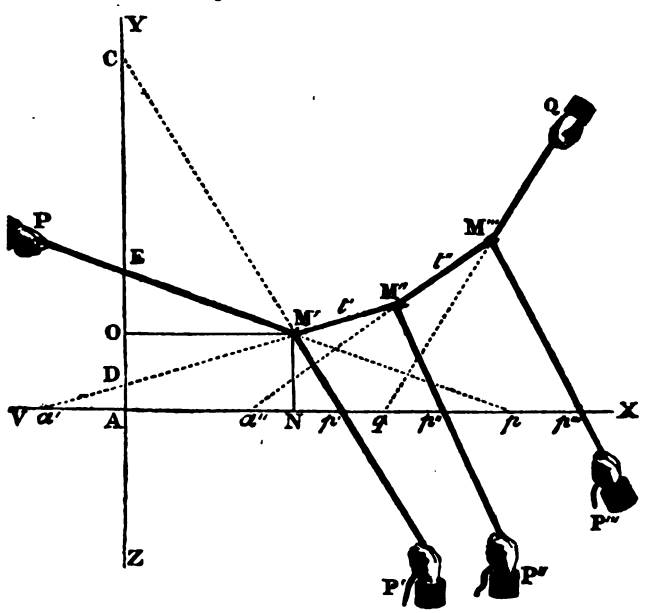
XLII.

On the Construction of the Funicular Polygon and the Catenary.

1. If any number of forces P, P', P'', P''', Q, (Fig. in page 316) acting in directions $PM', P'M', P'M'', P''M''', QM'''$[a] upon the cord $PM'M''M'''Q$, keep each other in equilibrio, then the figure into which the cord is bent by the action of these forces is called the *funicular polygon*; and since the *whole system* is in equilibrio, it is evident that every *part* of it must also be in equilibrio; thus P, P' and the *tension of the cord* $M'M''$ must be in equilibrio about the *knot* M'; *tension of the cord* $M''M'$, P', and *tension of the cord* $M''M'''$, about the *knot* M''; and, *tension of the cord* M'''

(a) These directions are supposed to lie all in the *same plane.*

$M'''M''$, P''', and Q, about the *knot* M'''. Take any point A, and through it draw the lines (or *axes*) VX, YZ at right angles to each other; produce PM', QM''' to meet VX in the points p, q; let VX cut the directions $P'M'$, $P''M''$, $P'''M'''$ in the points p', p'', p'''; and produce $M''M'$, $M'''M''$ to meet VX in the points $\alpha'\alpha''$. To facilitate the con-



struction of the polygon, let the angles Ppv, $M'p'V$, $M''p''V$, $M'''p'''V$, QqV (in which the direction of the forces P, P', P'', P''', Q are inclined to VX) be respectively denoted by the letters p, p', p'', p''', q; and the angles $M'\alpha'V$, $M''\alpha''V$ (in which the cords $M''M'$, $M'''M''$ are inclined to VX) by the letters α', α''; moreover, let the tensions of the cords $M'M''$, M''

$M''M'''$ in the *directions* $M'M'', M'M'''$ be represented by t', t'', in which case it is evident that the tension of those cords in the *opposite directions* ($M''M', M'''M''$) must be represented by $-t', -t''$.

2. Draw $M'N, M'O$ at right angles to AX, AY respectively; and produce $P'M', M''M'$ to meet AY in the points C, D. The effects of the forces P, P', t' (which keep the *knot* M' in equilibrio), when reduced to the *axis* AX, are measured respectively by $P \times \cos. p$, $P' \times \cos. p'$, $t' \times \cos. \alpha'$ [a]; and when reduced to the *axis* AY by $P \times \cos. M'EO$, $P' \times \cos. M'CO$, $t' \times \cos. M'DO$, or by $P \times \sin. p$, $P' \times \sin. p'$, $t' \times \sin. \alpha'$, which are equal to them [b]. But since the forces P, P', t', keep each other in *equilibrio* about the knot M', the *sum* of them, when estimated along the *same straight line*, must be equal to *nothing*; hence we have,

$$\left.\begin{array}{l} P \times \cos. p + P' \times \cos. p' + t' \times \cos. \alpha' = 0 \\ \text{and } P' \times \sin. p + P' \times \sin. p' + t' \times \sin. \alpha' = 0 \end{array}\right\} (A)$$

where these sines and cosines must be reckoned *positive*

(a) For P : part effective in direction AX (π) :: $M'p : pN$:: radius (1) : cos. p, $\therefore \pi = P \times \cos. p$; P' : part effective in direction AX (π') :: $M'p' : p'N$:: radius (1) : cos. p', $\therefore \pi' = P' \times \cos. p'$; t' : part effective in direction AX (T) :: $M'\alpha' : \alpha'N$:: radius (1) : cos. α', $\therefore T = t' \times \cos. \alpha'$; and in the same manner the *effective* part of the forces P, P', t' may be calculated for the *axis* AY.

(b) For cos. $M'EO$ = sin. $EM'O$ = sin. $M'pN$ = sin. p; cos. $M'CO$ = sin. $CM'O$ = sin. $M'p'N$ = sin. p'; and cos. $M'DO$ = sin. $OM'D$ = sin. $M'\alpha'N$ = sin. α'.

positive or *negative*, according as they are measured in the *same* or *opposite* directions upon the lines FX, FZ [*].

3. By referring the forces $-t^{(*)}$, P', t' (which keep each other in equilibrio about the knot M') to the axes FX, FZ, and resolving them in a similar manner, we should have,

$$\left.\begin{aligned} -t\times\cos.\alpha'+P'\times\cos.p'+t'\times\cos.\alpha'=0\\ \text{and}\ -t\times\sin.\alpha+P'\times\sin.p'+t'\times\sin.\alpha'=0\end{aligned}\right\}(B)$$

and extending the same process of reasoning to the three forces $-t'$, P'', Q (which are in equilibrio about the *last knot* M'') we have,

$$\left.\begin{aligned} -t'\times\cos.\alpha''+P''\times\cos.p''+Q\times\cos.q=0\\ \text{and}-t'\times\sin.\alpha''+P''\times\sin.p''+Q\times\sin.q=0\end{aligned}\right\}$$

4. The *law* therefore which obtains between the quantities composing these several equations is very evident; so that if there were *any number* of forces P', P'', P''' &c. between the *first* force (P) and the *last* (Q), we could easily estimate the relation between them and the *tensions upon the different parts of the cord adjoining to each knot*, at the time the equilibrium takes place; for

Let

(*) In these equations we have thought proper to make the signs of all these quantities *positive*; for it is very easy to restore to each quantity its *proper* sign, as will be seen when we come to the solution of an *actual example*.

(*) For this *negative* sign of t', see *the latter part of Art.* 1.

Let the *knots* in the cord be M', M'', M''' &c. . . $M'^{(n)}$ [a].
. . . . *forces* acting at them P', P'', P''' &c. . . $P'^{(n)}$
. . . . *angles* at which they act . . . p, p'', p''' &c. . . $p'^{(n)}$
. . . . *tensions* of $M'M'', M''M'''$ &c. t', t'', t''' &c. . . $t'^{(n)}$
. . . . *angles* at which t', t'', t''' &c. act, $\alpha', \alpha'', \alpha'''$ &c. . . $\alpha'^{(n)}$;

then the equations denoting the *equilibrium* at the several knots M', M'', M''' &c. will be as follow;

For the knot M'.

The equations marked (A).

For the knot M''.

The equations marked (B).

For the knot M'''.

$$\left.\begin{aligned} -t'' \times \cos.\alpha'' + P''' \times \cos.p''' + t''' \times \cos.\alpha''' = 0 \\ -t'' \times \sin.\alpha'' + P''' \times \sin.p''' + t''' \times \sin.\alpha''' = 0 \end{aligned}\right\} (C)$$

&c. &c. &c. &c. &c.

For the last *knot $M'^{(n)}$.*

$$\left.\begin{aligned} -t'^{(n-1)} \times \cos.\alpha'^{(n-1)} + P'^{(n)} \times \cos.p'^{(n)} + Q \times \cos.q = 0 \\ \text{and} -t'^{(n-1)} \times \sin.\alpha'^{(n-1)} + P'^{(n)} \times \sin.p'^{(n)} + Q \times \sin.q = 0 \end{aligned}\right\} (L).$$

5. By setting down these several equations in the form of A, $\overline{A+B}$, $\overline{A+B+C}$, &c. $\overline{A+B+C}$ &c. . . . $+L$, [b] we have,

$P \times$

[a] Here n denotes the *number of dashes at the letters*; M' standing at the *first* knot; M'' at the *second*; and so on.

[b] *i. e.* Set down (A) again; add the *first* lines of A and B together, and then the *second* lines; add the *first* lines of A, B, C together, and then the *second* lines; &c. &c.

$$\left.\begin{aligned} P\times\cos.p+P'\times\cos.p'+t'\times\cos.\alpha'=0\\ P\times\sin.p+P'\times\sin.p'+t'\times\sin.\alpha'=0\end{aligned}\right\}(A)$$

$$\left.\begin{aligned} P\times\cos.p+P'\times\cos.p'+P''\times\cos.p''+t''\times\cos.\alpha''=0\\ P\times\sin.p+P'\times\sin.p'+P''\times\sin.p''+t''\times\sin.\alpha''=0\end{aligned}\right\}(A+B)$$

$$\left.\begin{aligned} P\times\cos.p+P'\times\cos.p'+P''\times\cos.p''+P'''\times\cos.p'''+t'''\times\cos.\alpha'''=0\\ P\times\sin.p+P'\times\sin.p'+P''\times\sin.p''+P'''\times\sin.p'''+t'''\times\sin.\alpha'''=0\\ \text{\&c. \&c. \&c. \&c. \&c. \&c.}\end{aligned}\right\}(A+B\cdot$$

$$\left.\begin{aligned} P\times\cos.p+P'\times\cos.p'+\&c...P^{(n)}\times\cos.p^{(n)}+Q\times\cos.q=0\\ P\times\sin.p+P'\times\sin.p'+\&c...P^{(n)}\times\sin.p^{(n)}+Q\times\sin.q=0\end{aligned}\right\}(A+B+\&c..$$

6. On reviewing the equations in the preceding article, it may be observed that all their terms, *except the last*, consist of *known* quantities (viz. *the several forces* P,P',P'',P''' &c. *multiplied into the sines or cosines of the given angles in which their directions cut the line* VX). If, therefore, these known quantities in the *first* line of equation (A) be denoted by c, and in the *second* by s; and in the *succeeding* equations by $c',c'',$ &c. . . $c^{(n)}$, and s',s'', &c. . . $s^{(n)}$; then these equations may be transformed into the following ones; viz.

$$\left.\begin{aligned} c+t\times\cos.\alpha=0\\ s+t\times\sin.\alpha=0\end{aligned}\right\}(D)$$

$$\left.\begin{aligned} c'+t'\times\cos.\alpha'=0\\ s'+t'\times\sin.\alpha'=0\end{aligned}\right\}(E)$$

$$\left.\begin{aligned} c'+t''\times\cos.\alpha''=0\\ s'+t''\times\sin.\alpha''=0\\ \text{\&c. \&c. \&c. \&c.}\end{aligned}\right\}(F)$$

$$\left.\begin{aligned} c^{(n-1)}+Q\times\cos.q=0\\ s^{(n-1)}+Q\times\sin.q=0\end{aligned}\right\}(L').$$

7. From

7. From the equations marked *(D)*, we obtain,

$$t' \times \cos.\alpha' = -c, \quad \therefore \; t'^2 \times \overline{\cos.\alpha'}^2 = c^2,$$

$$\text{and } t' \times \sin.\alpha' = -s, \quad \therefore \; t'^2 \times \sin.\alpha'^2 = s^2,$$

$$\text{Hence } \frac{\sin.\alpha'}{\cos.\alpha'} = (\text{tang.}\,\alpha' =) \; \frac{s}{c},$$

$$\text{and } t'^2 \times \overline{\cos.\alpha'}^2 + \overline{\sin.\alpha'}^2 \; (= t'^2 \times \overline{\text{rad.}}^2 \text{ or } 1) \; = c^2 + s^2,$$

$$\therefore \; t' = \sqrt{c^2 + s^2};$$

and precisely in the same manner we may obtain the values of α'', α''' &c. q; and of t'', t''', t'''' &c. Q. We thus determine the *tension* and *position* of the several parts ($M'M''$, $M''M'''$, &c.) of the cord in succession, as well as *the magnitude and position of the last force Q, which keeps the whole in equilibrio.*

8. The manner in which the values of the several quantities t', t'', t''' &c, $\alpha', \alpha'' \alpha'''$ &c. are determined, corresponds therefore to the manner in which we may conceive the equilibrium to take place, in tracing the actions of the forces P, P', P'' &c. from one end of the cord to the other. For let us first suppose two forces P, P' (given both in *quantity* and *direction*) to act upon the *knot M'*, then from the equation *(D)* we find the quantity and direction of a *third* force t', which keeps these two in *equilibrio*; t' (or rather $-t'$) and P'' now become two given forces acting upon the *knot M''*, and from equation *(E)* we determine also the quantity and direction of a *third* force t'' which keeps *them*

them in equilibrio; and so we proceed till we arrive at the force Q, which acts at the *other end of the cord*. The conclusions here drawn are entirely independent of the *length of the cord*, or of the *relative length of its several parts*; which is perfectly consistent with the nature of the problem; for since the effects of forces acting in *straight lines* are the same at whatever points in those lines the forces act, it is evident that the sum total of the forces which thus keep each other in equilibrio will altogether depend upon the *angles* at which their directions are inclined to each other[a].

9. Suppose now all these forces except the *first and last* to act parallel to each other, and at right angles to the line AX[b] (as in the case of any number of *weights* P', P'', P''' &c. suspended by a cord $PM'M''M'''Q$ fastened to two hooks at the points

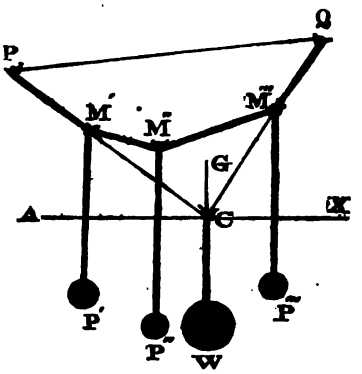
(a) As the forces P, P', P'' &c. are all supposed to act in the *same plane*, the directions of the *resulting forces* t', t'', t''' &c. will lie in that plane; and consequently the *cord itself*, after it is acted upon by these several forces and tensions, will be in that plane. If these forces do not act in the *same plane*, or if there be *several* forces acting in different planes upon the knots M', M'', M''' &c. then may they all be reduced to others whose quantities and directions may be estimated along the *axes* VX, YZ, according to the principles laid down in Sect. 12, page 74.

(b) As the point A may be *arbitrarily* assumed, take it in such manner that AX (drawn at right angles to $P'M', P''M''$, &c.) may pass through C the point of intersection of the directions PM', QM'' of the first and last force P and Q.

points P, Q), then will cos. $p' = \cos. p'' = \cos. p'''$ = &c. = 0, and sin. $p' = \sin. p'' = \sin. p'''$ = &c. = radius; hence the equations marked $(A + B + C \ldots L)$ in Art. 5, page 320, become

$$P \times \cos. p + Q \times \cos. q = 0, \quad \text{and}$$

$$P \times \sin. p + \overline{P' + P'' + P'''} \text{ &c.} \times \text{rad.} + Q \times \sin. q = 0.$$



From the nature of this problem it is evident that cos. p and cos. q will lie in different directions, $\therefore$ one will be *positive* and the other *negative;* and since the tensions upon P and Q are exerted *upwards*, and the forces P', P'', P''' &c. act *downwards*, P and Q must have the *same* sign, and $P' + P'' + P''' +$ &c. a *contrary* sign; these equations therefore may be transformed into

$$P \times \cos. p = Q \times \cos. q \ (G)$$

and $P \times \sin. p + Q \times \sin. q = \overline{P' + P'' + P'''} \text{ &c.} \times \text{rad.} \ (H)$.

From equation (G) we have $P : Q ::$ cos. q : cos. p :: cos. QCX : cos. PCA :: sin. QCG : sin. PCG :: sin. QCW : sin. PCW.

Substitute for Q in equation (H) its value found from equation (G) $\left(\text{viz. } \frac{P \times \cos. p}{\cos. q}\right)$, and we have

$$P \times \sin. p + \frac{P \times \cos. p \times \sin. q}{\cos. q} = \overline{P' + P'' + P''' + \text{&c.}} \times \text{rad.}$$

or

or $\frac{P \times \sin. p \times \cos. q + P \times \cos. p \times \sin. q}{\text{rad.}} = \overline{P' + P'' + P''' + \&c.} \times \cos. q,$

i. e. $P \times \sin. \overline{p+q}$ [a] $= \overline{P' + P'' + P'''} + \&c. \times \cos. q$;

$\therefore P : P' + P'' + P''' + \&c. :: \cos. q : \sin. \overline{p+q}$:: sin. QCW : sin. PCQ [b]; and in the same manner we might prove $Q : P' + P'' + P''' + \&c. ::$ sin. PCW : sin. PCQ.

10. On comparing these relations of P, Q and $P' + P'' + P'''$ &c. with what was shewn in Note (a) page 312, it appears that when any number of weights P', P'', P''' &c. are thus suspended from the hooks P and Q by means of *knots* in the cord $PM'M''M'''Q$, the *pressure upon the hooks P and Q* is the same as if a weight (W) equal to $P' + P'' + P'''$ &c. was suspended from the knot C in a cord PCG, whose length is determined by producing PM', QM''' till they meet in C; for according to the principles laid down in that Note, we have,

Pressure on P : pres. on $Q ::$ sin. QCW : sin. PCW;
Pressure on
$P : W (P' + P'' + P''' + \&c.) ::$ sin. QCW : sin. PCG;
and Pres. on
$Q : W (P' + P'' + P''' + \&c.) ::$ sin. PCW : sin. PCG;

$\therefore$ the pressures thus produced by the weight (W) bear the same relation to *each other* and to the *weight*

(a) See Plane Trigonometry, page 26.

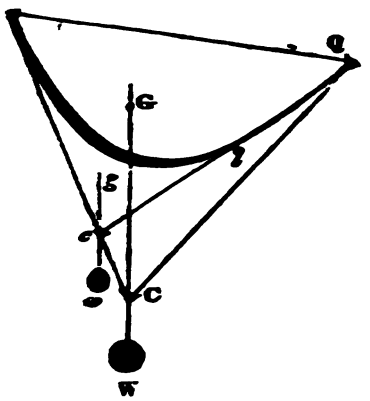
(b) For $PCQ = 180 - \overline{p+q}$, and sin. $\overline{p+q}$ = sine of its *supplement* $180^\circ - \overline{p+q}$.

weight (W) as the forces P, Q in the last article; the tensions of the cords PM', QM''' (i. e. the pressure upon the hooks P, Q) are consequently the *same* whether the weights P', P'', P''' &c. are suspended by knots in the cord at M', M'', M''' &c. or a weight *(W)* equal to *their sum* be suspended from C.

11. It is evident that the foregoing demonstration does not at all depend upon the *number* of the weights P', P'', P''' &c. suspended from the knots M' M'', M''' &c.; let us therefore suppose them to be increased *ad infinitum*, in which case the cord will assume the form of a *curve*, and the pressure upon the hooks P and Q will be the same as if a weight equal to $P' + P'' + P'''$ &c. was suspended by a cord whose length is PCQ, of which the *knot* C is determined by producing the *tangents* PM', QM''' till they meet in that point (Fig. in page 323). Or if we suppose the *cord itself* to have weight, then each particle of it may be considered as a weight suspended from and acting at the point to which it belongs, so that it will assume the form of a curve PqQ (Fig. in page 326); and its pressure upon the hooks P, Q will be the same as if a weight *(W) equal to the weight of the cord* was suspended from C; PC, QC being *tangents* to the curve in the points P, Q, and the *cord* PCQ *void of gravity*. It is further evident, that when once the equilibrium has taken place, the *tension at any point* q in the curve may be estimated in the same manner as

if

if the cord was suspended from that point. To find therefore the tension at the point q, we have only to draw the tangent qc meeting PC in c, and through c draw cw parallel to CW, and suppose w to be equal to the *weight of the part* Pq of the cord; then, because (by Art. 10) the pressures upon P or Q are to the *whole* weight (W) of the cord :: sin. QCW or sin. PCW : sin. PCQ, the *tension at* q will be to the weight (w) of *the part* Pq :: sin. Pcw : sin. Pcq, and *tension at* q : *pressure* at P :: sin. Pcw : sin. qcw. [b]

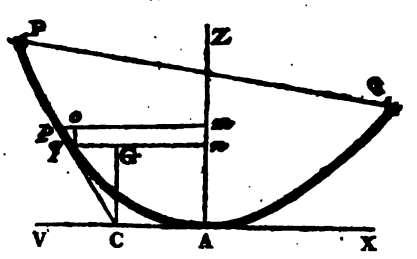


12. If

([a]) Since the pressure at P : W :: sin. QCW : sin. PCQ, the *pressure at* $P = \frac{W \times \text{sin. } QCW}{\text{sin. } PCQ}$; if $PCQ = 180°$, then sin. $PCQ = 0$, whilst W and QCW remain *finite*, $\therefore$ P in this case is *infinite*; which confirms what was said in Note (a), page 310, respecting the force necessary to keep a cord *perfectly straight* when supported at both ends, though acted upon only by its *own weight*.

([b]) Let G be the *center of gravity* of the cord, then its whole weight operates as if it was *concentrated* in G; but, from what has been here proved, it *also* operates as if it was concentrated in C; from hence it follows that G and C must lie in the same *vertical line*; i. e. "*if a line be drawn through* C (*the common intersection of the tangents* PC, QC) *at right angles to the horizon, it will pass through the center of gravity of the cord.*"

12. If the cord be of *uniform thickness*, and *perfectly flexible*, then it will not be difficult to find the *equation* to the curve whose form it will assume. For suppose it to be suspended from the points P and Q as before, and that the *equilibrium* has taken place; then, if A be the *lowest* point, it is evident that the *tangent* VAX will be *parallel to the horizon*[c]. Draw AZ at right angles to VX, and consider it as the *axis* of the curve, to which the perpendicular ordinates pm &c. are referred. Let $An = x$, $pn = y$, $Aq = z$, and draw qn indefinitely near to pm, and qo parallel to AZ, so that we shall have mn or $qo = \dot{x}$, $po = \dot{y}$; let the tangent to the point p (pC) meet VX in C, and draw CG parallel to AZ, in which case the triangles CGq, qop will be similar. Suppose now the *tension* at the given point A to be represented by the *constant quantity* (a); and since the cord is of *uniform thickness*, the weight of any *part* of it will be represented by the *length* of that part; hence the *weight* of the part qA may be represented by z; but (by Art. 11) the *tension at* A (a) : *weight of* qA



(c) For a tangent to the curve at any point between P and A is *inclined* to the horizon; and a tangent at any point between A and Q is *also* inclined to the horizon, but in an *opposite direction*; in passing therefore through the *lowest point* A it must become *parallel* to the horizon.

qA (z) :: sin. qCG : sin. qCA or sin. qCV or sin. CqG :: (by sim. Δ^s) sin. pqo : sin. qpo :: po : qo :: $\dot{y}$: $\dot{x}$; hence $a^2 : z^2 :: \dot{y}^2 : \dot{x}^2$, and a^2+z^2 : z^2 :: $\dot{x}^2+\dot{y}^2$ $(\dot{x}^2)$: $\dot{x}^2$, $\therefore \dot{x}=\frac{z\dot{z}}{\sqrt{a^2+z^2}}$; from which $x=\sqrt{a^2+z^2}+$ Cor.; but when $z=0$, $x=0$, $\therefore$ the *correct* fluent is $x=\sqrt{a^2+z^2}-a$; hence $a+x=\sqrt{a^2+z^2}$, and $a^2+2ax+x^2=a^2+z^2$, $\therefore z^2=2ax+x^2$, and $z=\pm\sqrt{2ax+x^2}$. Again, since $a : z :: \dot{y} : \dot{x}$, we have $\dot{y}=\frac{a\dot{x}}{z}=\frac{\pm a\dot{x}}{\sqrt{2ax+x^2}}$, whose *correct* fluent is $y=\pm a \times$ hyp. log. $\frac{a+x+\sqrt{2ax+x^2}}{a}$ $=\pm a \times$ hyp. log. $\frac{z+\sqrt{a^2+z^2}}{a}$ (a); and since the ordinate (y) has two *equal* values, the one *positive* and the other *negative*, it appears that the curve consists of two *similar branches* lying at equal distances on each side the line AZ. As this is the form which a *chain* consisting of very small links would *most readily* assume, this curve is called the *Catenary*, or *Catenarian curve*; but before we proceed to its *construction*, we shall exemplify the theorem which relates to the *funicular polygon*.

EXAM. I.

Suppose four forces P, P', P'', P''', which are to each other as the numbers 5, 4, 3, 2, *and whose directions are inclined to the line AX* (*see Fig. in page* 316) *in*

(a) For $a+x=\sqrt{a^2+z^2}$; and $\sqrt{2ax+x^2}=z$.

in angles of 30°, 40°, 50°, *and* 60° *respectively, to act at the same time upon the cord* $PM'M''M'''Q$ *by means of knots tied in the cord at the points* M', M'', M'''; *it is required to find the magnitude of a fifth force* (Q) *which will keep them all in equilibrio, and the form of the cord after the equilibrium has taken place.*

By referring to Art. 5th, 6th, and 7th of the present Section, we have,

$P = 5$ and $p = 30^0$
$P' = 4 \ldots p' = 40^0$
$P'' = 3 \ldots p'' = 50^0$
$P''' = 2 \ldots p''' = 60^0$

$\therefore c = P \times \cos. p + P' \times \cos. p' = 5 \times \cos. 30^0 + 4 \times \cos. 40^0 = 5 \times .8660 + 4 \times .7660 = 7.3940.$

$s = P \times \sin. p + P' \times \sin. p' = 5 \times \sin. 30^0 + 4 \times \sin. 40^0 = 5 \times .5000 + 4 \times .6427 = 5.0708.$

Hence $\tan. \alpha' = \frac{s}{c} = \frac{5.0708}{7.3940} = .6857 = \tan. 34^0. 26'$,

and $t' = \sqrt{s^2 + c^2} = 8.965.$

Also, $c' = c + P'' \times \cos. p'' = 7.3940 + 3 \times \cos. 50_0 = 7.3940 + 3 \times .6427 = 9.3221.$

$s' = s + P'' \times \sin. p'' = 5.0708 + 3 \times \sin. 50^0 = 5.0708 + 3 \times .7660 = 7.3688.$

Hence $\tan. \alpha'' = \frac{s'}{c'} = \frac{7.3688}{9.3221} = .7904 = \tan. 38^0. 19'$,

$t'' = \sqrt{s''^2 + c''^2} = 11.882.$

Lastly, $c'^{(b)} = c' + P''' \times \cos. p''' = 9.3221 + 2 \times \cos. 60^0 = 9.3221 + 2 \times .5000 = 10.3221.$

$s'' = s' + P''' \times \sin. p''' = 7.3688 + 2 \times \sin. 60^0 = 7.3688 + 2 \times .8660 = 9.1008.$

Hence $\tan. \varphi = \frac{s''}{c''} = \frac{9.1008}{10.3221} = .8816 = \tan. 41^0. 44'$,

$Q = \sqrt{s''^2 + c''^2} = 13.761.$

To

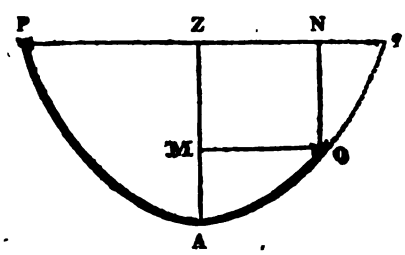
(b) Here c'' answers to $c'^{(n-1)}$ in equation (L') Art. 6, page 320; for (since there are *three* knots) $n - 1 = 2$.

To find the *angles* at the several knots M', M'' M''', we have $\angle PM'M'' = \alpha'M'p = 180^\circ - \overline{\alpha' + p}$ $= 180^\circ - \overline{34^\circ.26' + 30^\circ} = 115^\circ.34'$; $\angle M'M''\alpha'' = \alpha''$ $- \alpha' = 38^\circ.19' - 34^\circ.26' = 3^\circ.53'$, $\therefore$ $\angle M'M''M'''$ $= 176^\circ.7'$; $\angle M''M'''q = q - \alpha'' = 41^\circ.44' - 38^\circ.19'$ $= 3^\circ.25'$, $\therefore$ $\angle M''M'''Q = 176^\circ.35'$.

EXAM. II.

Suppose a perfectly flexible chain PAQ, of a given length (l), to be suspended from two hooks P, Q given in position; it is required to construct the catenary.

1. Draw PN parallel to the horizon, and QN at right angles to it; then, as the points P and Q are given in position, PN and QN may be considered as *given* quantities. Suppose A to be the *lowest* point, and draw AZ parallel to QN; draw also QM parallel to PN. Let $PN = b$, QN or $MZ = c$, $AZ = x$, $PZ = y$; then will NZ or $QM = PN - PZ = b - y$, and $AM = AZ - MZ = x - c$.



2. Let a = the *tension at the lowest point* A; then, by Art. 12, $PA = \sqrt{2ax + x^2}$, and AQ $= \sqrt{2a.\overline{x-c} + \overline{x-c}^2}$ [a]; hence

PA

(a) For since A is the *lowest point*, AZ is the *axis* of the curve, and the properties of the curve on each side of the axis are the

$$PA+AQ=\sqrt{2ax+x^2}+\sqrt{2a.\overline{x-c}+\overline{x-c}^2}=l,$$

$$y=a\times\text{hyp. log.}\frac{a+x+\sqrt{2ax+x^2}}{a},$$

and

$$b-y=a\times\text{hyp. log.}\frac{a+x-c+\sqrt{2a\times\overline{x-c}+\overline{x-c}^2}}{a}\text{ (b)},$$

are three *independent* equations, from which the values of the three *unknown* quantities a, x and y may be determined; when these equations are solved, the *length* and *position* of the axis AZ will be known; and as we shall also know the value of a, the whole curve may be constructed from the general equation between its abscissa and ordinate (viz. $y=a\times\text{hyp. log.}\frac{a+x+\sqrt{2ax+x^2}}{a}$, x and y being no longer AZ and PZ, but any abscissas or ordinates whatever.)

3. But it is evident, from the *transcendental* nature of these equations, that the values of a, x and y can only be obtained by a very tedious process of *approximation*; in the case therefore when P and Q are not placed in the *same horizontal line*, the construction of the catenary becomes a problem of very

the same, $\therefore AQ=\sqrt{2a\times AM+AM^2}$ from the *general property*, "$z=\sqrt{2ax+x^2}$."

(b) This equation also follows from the *general property*, "that "the *ordinate* $MQ=a\times\text{hyp. log.}\frac{a+AM+\sqrt{2a\times AM+AM^2}}{a}$."

very difficult solution; but if the points of suspension (P, q) *be* in the same horizontal line, then the curve may be constructed without much difficulty. For instance, Let the distance between the points of suspension ($Pq = 10$ feet, the length of the chain $PAQq = 20$ feet, then will $PZ = 5$ feet, and $PA = 10$ feet; the equation therefore to determine the value of a will be $\left(y = a \times \text{hyp. log.} \frac{z + \sqrt{a^2 + z^2}}{a}\right)$[a] $PZ = a \times \text{hyp. log.} \frac{PA + \sqrt{a^2 + PA^2}}{a}$, or $5 = a \times \text{hyp. log.} \frac{10 + \sqrt{a^2 + 100}}{a}$, $\therefore \frac{5}{a} = 2.3025$[b] $\times$ log. $\frac{10 + \sqrt{a^2 + 100}}{a}$; from which a, in a very few trials, is found to be equal to 2, 3 feet[c]. Substitute this value of a in the equation $z^2 = 2ax + x^2$ (where $z = PA$, $x = AZ$), then $x^2 + 4.6x = 100$, from which we obtain x or $AZ = 7.96$ feet. Having thus determined the *length* and *position* of the axis AZ, the curve may be constructed in the manner described in Art. 2.

(a) See Art. 12, page 328.

(b) For the *hyperbolic* logarithm of any number = 2.3025 × *common* logarithm of the same number.

(c) We thus obtain the *tension* at the point A in *feet*; but we should recollect that we have all along substituted the *length* of the chain for its *weight*; this 2, 3 *feet* therefore means that the tension at A is equal to the *weight* of 2, 3 feet of the chain.

Lecture XI.

ON THE

STRENGTH, STRESS, AND PRESSURE OF BEAMS.

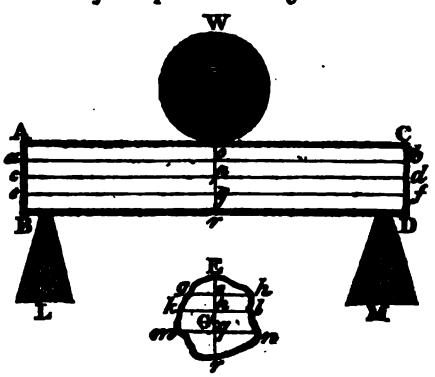
In estimating the strength and stress of beams, a variety of circumstances must be taken into consideration. Not only the *shape* and *position* of the beam, but the *force of adhesion* of the particles of which it is composed, must be brought into calculation. The problem, therefore, when taken in its fullest extent, is one of considerable difficulty; but if the beams are of *regular form* and *uniform texture*, it then becomes capable of solution upon mathematical principles. We shall begin with the case of *prismatic* beams supported horizontally at *both ends*.

XLIII.

On the stress of beams placed horizontally and supported at both ends.

1. Let *ABCD* represent a *longitudinal*, and *Ekrl* a *lateral* section, of a *prismatic* beam composed of parallel laminæ whose *lengths* are *ab*, *cd*, &c. *breadths* *gh*, *kl*, &c. and the *density* and *cohesion* of whose particles are perfectly *uniform* throughout. Suppose this beam to be supported at its two ends by the props *L*, *M*, and that on the *middle* of it there is placed the

the weight (W) touching it in the point E. From the point E draw Er, perpendicular to the horizon, and cutting the several laminæ in the points o, p, q &c.; it is evident that the pressure of the weight (W) will have a tendency to produce a *fracture* in the beam, beginning at the point r, and passing through the laminæ in succession till it arrives at the point E[a]. It is further evident that the tendency of the beam to *resist* fracture will depend partly upon the *adhesion* of its component particles, and partly upon the *distance* at which the *force* of adhesion acts from the point E, which point (as the fracture proceeds) may be considered as the *center of motion* of a lever, at the extremities of whose arms Er, Eq, Ep &c. this force is applied. Let σ = the uniform adhesive force of the particles of the beam, then the adhesive force of *the particles composing the line* gh will be measured by $\sigma \times gh$, and the *distance* at which it acts from E is Eo, $\therefore$ the *whole* force of the line of particles



(a) It is almost impossible to represent distinctly a *longitudinal* and a *lateral* section of the beam in the *same figure;* the reader therefore must conceive that the plane of the figure $Emrl$ is placed at right angles to the plane of the figure $ABCD$, in such manner that E may become the point *where* W *touches the beam.* and the line Er *the common intersection of the two planes.*

particles (gh) to resist fracture$=\sigma \times gh \times Eo$; for the same reason the force of the particles in the line $kl=\sigma \times kl \times Ep$; and in the line $mn=\sigma \times mn \times Eq$; &c. &c. hence the whole force of the beam to resist fracture $=\sigma \times \overline{gh \times Eo + kl \times Ep + mn \times Eq + \&c.}$ = (if G be the *center of gravity* of the figure $Ekrl$) $\sigma \times \overline{gh + kl + mn + \&c.} \times EG = \sigma \times$ *area of figure* $Ekrl \times EG$.

2. If σ be *given* (i.e. in *different beams* of the same length cut out of the *same species of matter*) this tendency to resist fracture will vary as the *area of a lateral section* × the *distance of its center of gravity from the point where the weight acts.* In *cylinders* and *square prisms* (and in all cases where the *sections* of the prisms are *similar* figures) this resistance therefore will vary as the *cubes* of their depths or homologous sides; and in beams whose sections are of an *oblong* form, it will vary as the *breadth* × *square of the depth.*[b] If the *area of the section* be also given, then $R \propto$

([b]) For let Figs. 1, 2, 3 represent respectively a section of a *cylinder, squar prism,* and right-angled *parallelopipedon,* then in Fig. 1, *area* $EArB \times EG \propto AB^2 \times \frac{1}{2}AB \propto AB^3$; in Fig. 2, *area* $ECrD \times EG \propto CD^2 \times \frac{1}{2}CD \propto CD^3$; and in Fig. 3, *area* $EFrK \times EG \propto FK \times Er \times \frac{1}{2}Er \propto FK \times Er^2$.

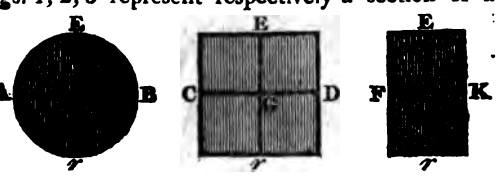


Fig. 1. *Fig. 2.* *Fig. 3.*

$R \propto EG$; i. e. if the *same* prismatic beam be placed horizontally *with its sides in different positions*, then its tendency to resist fracture will vary as *the distance of the center of gravity of a lateral section of it from the point where the weight acts*, and will be *greatest* when it is placed with the *narrower part* of the prism *uppermost*[a].

3. The foregoing expressions apply only to beams of the *same length* whose *own weights* are not taken into the consideration. Suppose now L, l to represent the *lengths*, and W, w the *weights* of two prismatic beams supported respectively at both ends; then W, w may be considered as collected in the *centers of gravity* of the beams, and acting at

(a) Let *ABCD* (Fig. 1.) represent a section of an *oblong* beam; then its tendency to resist fracture when *its side AC* is uppermost (or when placed *edgeways*) : the tendency when *AB* is uppermost (or when placed *flatways*) :: $EG : Eg :: \frac{1}{2}AB : \frac{1}{2}AC :: AB, AC$, G and *g* being the position of the *center of gravity* when the beam is placed in these respective positions. In Fig. 2, where the section is a *triangle*, these tendencies to resist fracture are as ($EG : Eg$::) $\frac{2}{3}EC : \frac{1}{2}EC :: 2 : 1$. In Fig. 3, where the section is a *Parabola*.

Fig. 1.

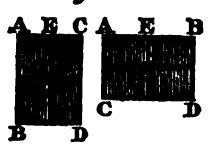


Fig. 2.



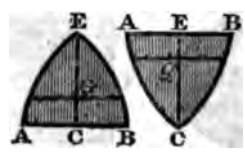


Fig. 3.

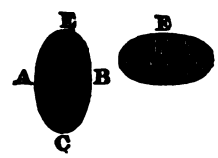


Fig. 4.

at the distance $\frac{1}{2}L$, $\frac{1}{2}l$ from the points of support; hence the stress or tendency *to produce fracture* in each beam will be measured by $\frac{1}{2}L \times W$ and $\frac{1}{2}l \times w$. But it is evident that the *whole strength* of the beams will be *directly* as the tendency to *resist* fracture, and *inversely* as the tendency to *produce* fracture; if therefore we denote their relative *strengths* by S,s; the *areas of their sections* by A,a; and the *depths of the centers of gravity* of the sections by G,g; we shall have (in the case of beams supporting only their *own weights*) $S : s :: \frac{A \times G}{\frac{1}{2}L \times W} : \frac{a \times g}{\frac{1}{2}l \times w} :: \frac{A \times G}{L \times W} : \frac{a \times g}{l \times w}$.

4. Let D,d represent the *diameters* of two *cylindric* beams, or the *sides* of two beams in the form of *square prisms*, then (by Art. 2) $S : s :: \frac{D^3}{L \times W} : \frac{d^3}{l \times w}$; and if they at the same time be *similar* (in

Parabola, they are as ($EG : Eg$::) $\frac{1}{4}EC : \frac{1}{4}EC :: 3 : 2$. And in Fig. 4, where the section is an Ellipse, they are as ($EG : Eg$::) $EC : AB$:: *axis major* : *axis minor*.

By a similar process, when the section of the prism is a *square* or an *oblong*, we might compare its strength when its *angular point* is uppermost with its strength when a *side* is uppermost; thus, suppose a *square* prism to be placed first with its *diagonal* and afterwards with its *sides* in a vertical position, then its strength in the *former* case : its strength in the *latter* :: the *diagonal* : the *side* of the square :: $\sqrt{2} : 1$, for the *depth of the center of gravity* in the former case = *half the diagonal*, and in the latter = *half the side*.

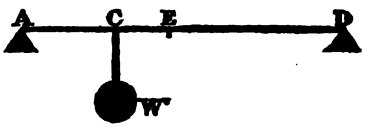
(in which case $L : l :: D : d$, and $W : w :: D^3 : d^3$)[a] $S : s :: \frac{D^3}{D^4} : \frac{d^3}{d^4} :: \frac{1}{D} : \frac{1}{d}$, or $\frac{1}{L} : \frac{1}{l}$. Hence it appears that the strength of *similar cylindric* or (*square*) *prismatic* beams varies *inversely* as their *linear dimensions*; i.e. whilst the linear dimensions of beams of this form are *increased*, their *strength* (so far as it is counteracted by the *increase of their own weight*) is continually *diminished*.

5. But if, besides their *own weights*, these beams be made to support *other weights* W', w' placed at their *middle points*, then it is evident that their *tendency to fracture* will be *increased*; and *that* in such a manner as to make (in the *general* case) $S : s :: \frac{A \times G}{L.\overline{W + W'}} : \frac{a \times g}{l.\overline{w + w'}}$, and (in the case of *cylinders* and *square prisms*) $S : s :: \frac{D^3}{L.\overline{W + W'}} : \frac{d^3}{l.\overline{w + w'}}$; for since these weights act at the *same distance* from the points of support with W, w, the tendency to fracture in *one* case will be measured by $\frac{1}{2}L.\overline{W + W'}$, and in the *other* by $\frac{1}{2}l.\overline{w + w'}$. If the weights of the *beams* be so small, when compared with the weights *supported*, as to make it unnecessary to take them into the consideration,

(a) For the *weights* of bodies of the *same density* are as their *solid contents*; if therefore the bodies be *similar* solids, their *weights* will be as *the cubes of their linear dimensions*.

ration, then $S : s :: \frac{A \times G}{L \times W'} : \frac{a \times g}{L \times W'}$ or $\frac{D^3}{L \times W'} : \frac{d^3}{l \times w'}$.

6. Hitherto we have estimated the strength of beams only at the *middle points* between the two props. In order to ascertain their strength at *any other point*, we must find how the *stress* varies at different parts of the same beam. Let us first take the case of a weight (W), supported at the point (C), by the beam AD, whose *own* weight is not taken into the consideration. Then (see Note *a*, page 232) the pressure upon the *prop* $A = \frac{W' \times CD}{AD}$, and upon the *prop* $D = \frac{W' \times AE}{AD}$; ∴ the *stress at* C, as estimated from the *prop* A, will be measured by $\frac{W' \times CD \times AC}{AD}$; and estimated from the *prop* D, it will be $\frac{W' \times AC \times CD}{AD}$[b]; hence the *whole stress* at the point $C = \frac{2W' \times AC \times CD}{AD}$, which



(b) For the stress at C, *as estimated from the prop* A, may be considered as produced by the weight $\frac{W' \times CD}{AD}$ acting *at the extremity of a lever whose length is* AC; and the same for the stress as *estimated from the prop* D.

which *varies* as $AC \times CD$, since $\frac{2W'}{AD}$ is a *given* quantity.

7. For the purpose of estimating the stress of a beam at any point C arising from its *own weight*, let $ABCD$ represent a *section* of the beam cut through the axis of the prism, and let $Ac = x$, $cd = \dot{x}$,



W = weight of the beam; then the *weight* of the indefinitely small part $cefd$ of the prism $= \frac{W \times \dot{x}}{AD}$, and its *pressure upon the prop* F (= its *weight* $\times \frac{Ac}{AD}$) $= \frac{W \times x\dot{x}}{AD^2}$. Hence the *stress* at c (arising from the weight of the indefinitely small part $cefd$) estimated from $D = \frac{W \times cD \times x\dot{x}}{AD^2}$; but the stress at c : stress at C :: cD : CD, $\therefore$ the stress at C arising from the weight of $cefd$ $= \frac{W \times CD \times x\dot{x}}{AD^2}$, whose *fluent* is $\frac{W \times CD \times x^2}{2AD^2}$, and when $x = AC$, it is $\frac{W \times CD \times AC^2}{2AD^2}$; this *latter* quantity therefore expresses the stress at C arising from *the weight of the part* AC *of the beam*[a]. For the

(a) For the stress at C *arising from the part* AC *of the beam* is equal to the *sum of all the* $\frac{W \times CD \times x\dot{x}}{AD^2}$ contained between A and C, i. e. to *the fluent of* $\frac{W \times CD \times x\dot{x}}{AD^2}$ when $x = AC$.

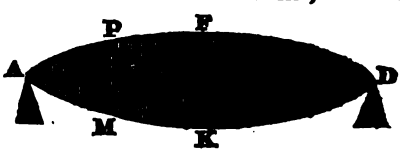
the same reason, the stress at C arising from *the weight of the part* $CD = \frac{W \times AC \times CD^2}{2AD^2}$; hence the *whole stress* at $C = \frac{W \times CD \times AC^2 + W \times AC \times CD^2}{2AD^2}$ $= \frac{W \times AC \times CD \times \overline{AC + CD}}{2AD^2} = \frac{W \times AC \times CD}{2AD}$, which *varies* as $AC \times CD$ as before. Whether therefore the *weight of the beam* be or be *not* taken into the consideration, the stress at any point C *varies* as $AC \times CD$.

8. Since $AC \times CD$ is *greatest* when $AC = CD$, the *stress* will be greatest at the *middle point* E; and the stress at this *middle point* will be to the stress at *any other point* $C :: \frac{1}{2}AD \times \frac{1}{2}AD$ $(\frac{1}{4}AD^2) : AC \times CD$. But the *strength* at any point of a *prismatic* beam will be *inversely as the stress*[b], $\therefore$ strength at the *middle point* E : strength at *any other point* $C :: AC \times CD : \frac{1}{4}AD^2$; hence the strength of the beam keeps continually increasing as we advance from the *middle point* towards either of the props; if therefore the greatest weight which can be sustained at the *middle point* be called (w), a weight which is greater than (w) in the ratio of $\frac{1}{4}AD^2 : AC \times CB$ may be

(b) For the strength varies as the *tendency to resist fracture* directly and *stress* inversely; but in a *prismatic* beam the tendency to resist fracture is the *same* at every point, $\therefore$ the strength varies *inversely as the stress*.

be sustained at C; call this weight (W), then $W : w :: \frac{1}{4}AD^2 : AC \times CD$, or $W = \frac{w \times AD^2}{4AC \times CD}$ [a].

9. As the stress upon a *prismatic* beam is greatest at the *middle* point, and *decreases* from the middle towards the ends, it is evident that for a beam *to be equally strong throughout* it must be of a *tapering* form [b]; and if the sides of such a beam are *parallel planes*, it is not difficult to shew that a *longitudinal* section of it must be an *ellipse*. For let the curve $APDM$ (whose *axis* is AD) represent this longitudinal section, and let a = the *thickness* or *breadth* of the beam; then a *lateral* section of the beam at any point C will be an *oblong* whose *breadth* is a, *depth* PM, and the depth of its *center of gravity* PC



(a) We thus determine the weight which might be sustained at some *given point* C; if the *weight* were given, and it was required to determine the point C where it could be sustained, then let $AC = x$, $AD = a$, and $\therefore CD = a - x$; substitute these values for AC, AD, CD, and we have $W : w :: \frac{1}{4}a^2 : ax - x^2$, $\therefore W \times \overline{ax - x^2} = \frac{1}{4}a^2 w$, which equation solved gives $x = \frac{a}{2} \pm \frac{a\sqrt{W-w}}{2\sqrt{W}}$, where x has *two* values, as there evidently will be *two* points equally distant from the *middle* point where a given weight may be sustained.

(b) By Art. 1, the tendency to resist fracture varies as *area of section* $\times EG$; $\therefore$ for the strength to be the *same* throughout, *area of section* $\times EG$ must decrease as the *stress* decreases.

PC or $\frac{1}{2}PM$; hence (by Art. 2) the tendency of the beam to resist fracture at any point C is as $a \times PM^2$; but the *stress* at C is as $AC \times CD$; $\therefore$ *strength* at C is as $\frac{a \times PM^2}{AC \times CD}$ or as $\frac{PM^2}{AC \times CD}$ a being constant; hence if $PM^2 \propto AC \times CD$, the strength will be the *same* at every point; but in this case the curve $APDM$ is an *ellipse* whose *axis major* is AD, and *axis minor* PK (the depth of the beam at the middle point E)[c].

10. Before we can apply the foregoing expressions to finding the *actual* strength of bars or beams, it is necessary that we should have some *standard* of strength to refer them to. In *Emerson's* Mechanics, and in other works where this subject is treated more at large, the reader will find *tables* in which the relative strength of a variety of substances is expressed in numbers. From these tables we shall select only *two* instances, which will be quite sufficient for the purpose of explaining the manner in which problems of this kind may be solved. The *first* is, "That a small beam of *oak*, a *foot* "long

(c) A beam of the form of either of the *semi-ellipses* $APDC$, $ACDM$ will *also* be equally strong throughout; for the depth of the *center of gravity* of the lateral section in these cases will be $\frac{1}{2}PC$ or $\frac{1}{2}CM$, and $\therefore$ the *strength* will vary as $\frac{PC^2 \text{ or } CM^2}{AC \times CD}$ as before.

" long and *one inch* square, is able (when supported " at both ends) to sustain at its *middle point* " a weight of 600 pounds." The *second*, " that a bar " of *iron* of the same dimensions would sustain in " the same manner a weight of 2090 pounds." It is also supposed that the beam of oak weighs *half* a pound, and the bar of iron *three* pounds.

EXAM. I.

What weight might be sustained at the middle point of a prismatic beam of oak, whose length is 6 *feet, and its end* 4 *inches square.*

Let S = the strength of the beam required, s = the strength of a beam whose length is *one foot*, and square end *one inch*; W = weight of the *larger* beam, w = *half* a pound; $L = 6, l = 1$; $D = 4, d = 1$; W' = weight required, $w' = 600\,lbs$; then, by Art. 5,

1. If the *weight of the beams* be not taken into the consideration, $S : s :: \frac{D^3}{L \times W'} : \frac{d^3}{l \times w'} :: \frac{4^3}{6 \times W'} : \frac{1^3}{1 \times 600}$; but from the conditions of the problem $S = s$[a], $\therefore \frac{4^3}{6 \times W'} = \frac{1^3}{1 \times 600}$ or $W' = 6400\,lbs$.

2. If the weight of the beams *be* taken into the consideration, then $S : s :: \frac{D^3}{L.W + W'} : \frac{d^3}{l.w + w'}$::

(a) Since the beams are supposed to be just capable of sustaining the weights W', w', their *strength* as compared with those weights must be the *same*.

$:: \frac{4^3}{6.48+W'}$ [b] $: \frac{1^3}{1.\frac{1}{2}+600}$, $\therefore \frac{64}{6.48+W'} = \frac{1}{600\frac{1}{2}}$,
or $W' = 6357\frac{1}{3}$ *lbs.*

EXAM. II.

Supposing its own weight not to be taken into the consideration, what must be the depth of a beam of an oblong prismatic form, whose breadth is 2 *inches and length* 6 *feet, to support a weight of* 6400 *lbs.?*

Let B=*breadth* of the beam required, and b=breadth of the beam whose length is *one foot* and square end *one inch;* then retaining the notation of the preceding example, and referring to Note (*a*), p. 335, and Art. 5, we have $S : s : \frac{B \times D^2}{L \times W'} : \frac{b^3}{l \times w'} :: \frac{2 \times D^2}{8 \times 6400} : \frac{1^3}{1 \times 600}$, $\therefore$ (when $S=s$) $2D^2 = 85.3333$, or $D = 6.53$ inches [c].

EXAM. III.

What weight might be supported at the middle point of a bar of iron 10 *feet long, and the side of whose*

(b) The *weights of the two beams* are to each other as their *solid contents*; $\therefore W : w\ (\frac{1}{2}) :: L \times D^2 : l \times d^2 :: 8 \times 16 : 1 \times 1 :: 96 : 1$, or $W = 48$ *lbs.*

(c) The *length* of this beam being the same with that in the preceding example, their *solid contents* will be as the *areas of their lateral sections*, i.e. the beam in Exam. 1st : that in Exam. 2d $:: 4^2 : 2 \times 6.53 :: 16 : 13.06$; the *oblong* beam therefore placed *edgeways* is as strong as the *square* one, although it contains *only about* $\frac{13}{16}$*ths of its substance.*

whose square end is 3 *inches, supposing its own weight to be taken into consideration?*

Here $S:s::\frac{D^3}{L.\overline{W+W'}}:\frac{d^3}{l.w+W'}::\frac{3^3}{10.270+W'}$ [a]

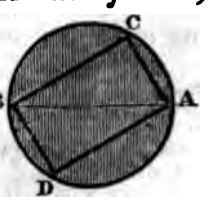
$:\frac{1^3}{1.3+2190}$, $\therefore$ (when $S=s$) $W'=\frac{2193\times 27}{10}-270$

$=5651.1$ *lbs.*

EXAM. IV.

Find the breadth and depth of the strongest oblong beam which can be cut out of a given Cylinder.

Let $ACBD$ represent a *section* of the cylinder, whose diameter $AB=a$; and suppose AC to be the *breadth* of the beam required, and BC its *depth*; let $AC=x$, then $BC^2=AB^2-AC^2=a^2-x^2$; but by Art. 2, the *strength*



of this beam will vary as $AC\times BC^2\propto x\times\overline{a^2-x^2}$; hence $a^2x-x^3=$ max.; $\therefore\ a^2\dot{x}-3x^2\dot{x}=0$, and $x=\frac{a}{\sqrt{3}}=AC$; $BC=\sqrt{a^2-\frac{a^2}{3}}=a\sqrt{\frac{2}{3}}$; the *strongest* oblong beam therefore which can be cut out of a given *cylinder* is that whose *breadth* $=\frac{a}{\sqrt{3}}$ and *depth* $=a\sqrt{\frac{2}{3}}$.[b]

Ex.

(a) Here $W:w$ (3) $::L\times D^3:l\times d^3::10\times 9:1\times 1$, $\therefore W=270$ *lbs.*

(b) Hence AB^2, BC^2, CA^2 are to each other as a^2, $\frac{2a^2}{3}$, $\frac{a^2}{3}$, or as 3, 2, 1; the *strongest oblong beam* therefore which can be cut out

EXAM. V.

What weight will a beam of oak, whose length is 8 *feet and side of its square end* 3 *inches, sustained at the distance of* 3 *feet from one of the props?*

By a calculation similar to that of Case 1, Exam. 1, it will be found that a beam of these dimensions would sustain a weight of 2025 *lbs.* at its *middle* point; ∴ by Art. 8, the weight which it would sustain at the distance of 3 feet from the prop A (Fig. in page 339) $= \left(\frac{w \times AD^2}{4AC \times CD} = \right) \frac{2025 \times 64}{4 \times 3 \times 5} = 2160\ lbs.$

EXAM. VI.

Find the distance from one of the props, at which 5 ***tons weight might*** *be sustained upon a bar of iron whose length is* 10 *feet, and the side of whose square end is* 3 *inches.*

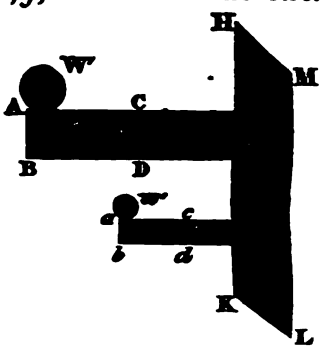
From Exam. 3, it appears that this bar would sustain at its *middle* point 5651.1 *lbs*; ∴ (see Note (*b*), page 341) the distance from the prop A at which it would sustain 5 tons (or 11200 *lbs*) $= \left(\frac{a}{2} - \frac{a\sqrt{W-w}}{2\sqrt{W}} = \right) \frac{10}{2} - \frac{10\sqrt{11200-5651.1}}{2\sqrt{11200}}$
$= 5 - 3.52 = 1.48$ feet.

out of a *given cylindric one*, is that in which the *squares* of the *diameter* of the cylinder, of the *depth* of the beam, and of its *breadth*, are to each other as the numbers 3, 2, 1.

XLIV.

On the strength and stress of beams supported at one end.

1. Let $ABEF$, $abef$ represent the longitudinal sections of two prismatic beams fixt horizontally into the wall $HKLM$; then the tendency of these beams to *resist* fracture at the ends EF, ef, where they are inserted into the wall, will be measured in the same manner as in the preceding cases (viz. by the *area of a lateral section* × *depth of its center of gravity*), except that in this case the fracture will begin at the *upper* points F, f, and end at the *lower* points E, e. The tendency to *produce* fracture will be the *weight* of the beams acting at the distance of their centers of gravity from the ends EF, ef. Retaining therefore the notations of articles 3, 4, 5, pp. 337, 338, and putting S for the *strength of the beam* $ABEF$, and s for *the strength of the beam* $abef$, we have $S : s :: \frac{A \times G}{\frac{1}{2}L \times W} : \frac{a \times g}{\frac{1}{2}l \times w}$; or if any weights W', w' are placed



(a) Since $S : s \left(:: \frac{A \times G}{\frac{1}{2}L \times W} : \frac{a \times g}{\frac{1}{2}l \times w} \right) :: \frac{A \times G}{L \times W} : \frac{a \times g}{l \times w}$, these expressions accord with those of Art. 3, page 338; the same conclusions therefore may be drawn with respect to *cylindric*

placed at the *other ends* of the beams, then (since the effects of these weights to produce fracture will be measured by $W' \times L$ and $w' \times l$) we shall have (as in Art. 5, p. 338) $S : s :: \frac{A \times G}{L \cdot \frac{1}{2}W + W'} : \frac{a \times g}{l \cdot \frac{1}{2}w + w'}$; moreover, if the weights W, w of the *beams* be *very small* when compared with the weights W', w', then $S : s :: \frac{A \times G}{L \times W'} : \frac{a \times g}{l \times w'}$.

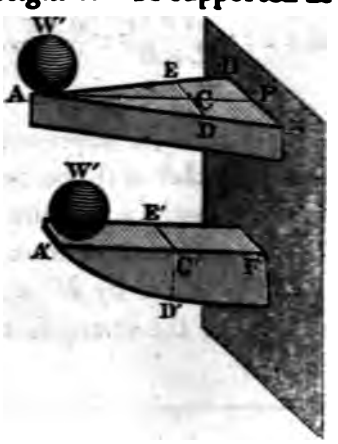
2. Let W, w represent the weights of *the parts* $ABDC, abdc$ of the beams, then the tendency of those parts to produce fracture at C, c will be measured by $\frac{1}{2}W \times AC$ and $\frac{1}{2}w \times ac$; and the tendency of the weights W', w' by $W' \times AC$ and $w' \times ac$; $\therefore$ if S, s represent the strength of the beams at the points C, c,

dric and *similar prismatic* beams, as in Art. 4; viz. that $S : s :: \frac{D^3}{L \times W} : \frac{d^3}{l \times w}$ in *one* case, and as $\frac{1}{L} : \frac{1}{l}$ in the *other*.

Also, since in the *same* beam placed with its sides in different positions, $S : s :: G : g$, the *same comparison* may be made as to the relative strength of beams placed *flatways* or *edgeways* as was made in Note (*a*), page 336; except that (since in this case the fracture begins at the *upper* side instead of the *lower*, and since the depths of the center of gravity are measured *from the points where the fracture terminates*) in beams whose lateral sections are *triangular* or *parabolic*, the ratio of $G : g$ (and consequently of $S : s$) will be that of $CG : cg$, instead of $EG : Eg$. From hence it appears, that when beams are supported *only at one end*, such as are of *triangular* or *parabolic* form will be stronger when their *broader* side, that when their *narrower* side, is uppermost; the *former* in the ratio of 2 : 1, the *latter* in the ratio of 3 : 2.

C, c, then $S : s :: \frac{A \times G}{AC \cdot \frac{1}{2}W + W'} : \frac{a \times g}{ac \cdot \frac{1}{2}w + w'}$; or if W, w be *very small* when compared with W', w', then $S : s :: \frac{A \times G}{AC \times W'} : \frac{a \times g}{ac \times w'}$

3. Hence, if a *given weight* W' be supported at the end of a given beam whose weight is so small as not to be taken into the consideration, the strength of that beam to support the weight W' at any point C between A and F will vary as $\left(\frac{A \times G}{AC \times W'} \text{ or as}\right) \frac{A \times G}{AC}$.



Let the beam be in the form of an *isosceles wedge* whose flat sides are parallel to the horizon, and whose given *depth* $= d$, then $A = ED \times d$, and $G = \frac{1}{2}d$, $\therefore A \times G = \frac{1}{2}ED \times d^2 \propto ED \propto EC$ which $\propto AC$[a]; hence the strength $\propto \frac{AC}{AC}$, or is *constant;* a beam of this form therefore will be *equally strong* throughout.

(a) ED is drawn *parallel* to the base of the wedge, and AF *perpendicular* to it; $\therefore AF : AC :: HK : ED = \frac{AC \times HK}{AF} \propto AC$ (HK and AZ being *constant*); hence EC ($= \frac{1}{2}ED$) varies also as AC.

throughout. Or if the sides of the beams be *parallel planes*, and its longitudinal section a *semi-parabola*, as $A'C'F'D'$; then let d = the given *breadth*, and we have $A = d \times C'D'$, $G = \frac{1}{2}C'D'$, $\therefore A \times G = \frac{1}{2}d \times C'D'^2 \propto C'D'^2$; hence $S \propto \frac{C'D'^2}{A'C'}$; but since $A'D'$ is a *parabola*, $A'C' \propto C'D'^2$; a beam therefore of this form will *also* be equally strong throughout.[b]

4. Suppose the beams in Art. 1. to be *cylinders* or *square prisms*, whose diameters or sides are D, d, then $S : s :: \frac{D^3}{L.\frac{1}{2}W + W'} : \frac{d^3}{l\frac{1}{2}w + w'}$; let $w' = 0$, then $S : s :: \frac{D^3}{L.\frac{1}{2}W + W'} : \frac{d^3}{\frac{1}{2}l \times w}$, in which case $S : s$ expresses the *relative strength* of two cylindric beams whose *lengths* are L, l; *diameters* D, d; *weights* W, w; the *former* of which supports the given weight W' at the *end* of it, and the latter supports only *its own* weight.

5. Let $d = D$, then $S : s :: \frac{1}{L.\frac{1}{2}W + W'} : \frac{1}{\frac{1}{2}l \times w} ::$ $\left(\text{for } w = \frac{W \times l}{L}\right)^{[c]} \frac{1}{L.\frac{1}{2}W + W'} : \frac{2L}{W \times l^2}$; in this case,

(b) If a longitudinal section of the beam be a *whole parabola*, then will it also be *equally strong throughout*; for in this case $G = C'D'$, and $\therefore A \times G \propto C'D'^2$.

(c) Since $d = D$, the *weights* of the beams must be as their *lengths*, $\therefore w : W :: l : L$, or $w = \frac{W \times l}{L}$.

case, therefore, $S : s$ expresses the relative strength of two cylindric beams of the *same diameter*, *one* of which supports the given weight W' at its end, and the other supports only its *own weight*; and if the beam whose length is L *breaks* when W' is placed at the end of it, the beam whose length is l will *break with its own weight*; hence let $S = s$, or $\frac{1}{L.\frac{1}{2}W + W'} = \frac{2L}{W \times l^2}$, then $l^2 = \frac{L^2.W + 2W'}{W}$, and $l = L\sqrt{\frac{W + 2W'}{W}}$ = *the length of the beam which would break by its own weight.*

6. Let the beams be *similar* cylinders, then $D^3 : d^3 :: L^3 : l^3$, $\therefore S : s \left(:: \frac{L^3}{L.\frac{1}{2}W + W'} : \frac{l^3}{\frac{1}{2}l \times w} \right) :: \frac{L^2}{\frac{1}{2}W + W'} : \frac{l^2}{\frac{1}{2}w} :: \left(\text{for } w = \frac{W \times l^3}{L^3} \text{ (a)} \right) \frac{L^3}{\frac{1}{2}W + W'} : \frac{2L^3}{W \times l}$; and when $S = s$, $\frac{L^2}{\frac{1}{2}W + W'} = \frac{2L^3}{W \times l}$, $\therefore \frac{1}{2}W + W' = \frac{W \times l}{2L}$, or $l = \frac{L.W + 2W'}{W}$; if therefore a cylindric beam whose length is L *breaks* with the given weight W' placed at the end of it, a *similar* cylindric beam whose length is $\frac{L.W + 2W'}{W}$ will break with its *own weight*.

7. From

(a) The weights of *similar* cylinders of the same density are as the *cubes* of their *diameters* or *lengths*, $\therefore w : W :: l^3 : L^3$, or $w = \frac{W \times l^3}{L^3}$.

7. From Art. 7, page 340 (See also the *Figure* in that page), it appears that if a beam be supported at *both* ends, then the *stress at any point C* $= \frac{W \times AC \times CD}{2AD}$; hence the stress at the *middle* point $= \frac{W \times \frac{1}{4}AD^2}{2AD} = \frac{1}{8}W \times AD$; suppose a beam of the same diameter and only *half* the length to be supported at *one* end, then the stress at the supported end = (the weight of the beam × distance of its center of gravity from the end =) $\frac{1}{2}W \times \frac{1}{4}AD$ $= \frac{1}{8}W \times AD$; the stress therefore at the *middle* point of a beam supported at *both* ends is equal to the stress of a beam of the same diameter and of *half* the length supported only at *one* end; and, consequently, if a beam of any length would break by its *own weight* when supported at *both* ends, a beam of the same diameter and of *half* that length would break by its own weight when supported only at *one* end. The expressions deduced in Art. 5 and 6 will therefore apply to beams of *double* the length when supported at *both* ends.

EXAM. I.

What must be the length of a beam of oak one inch square, supported at both ends, which is just capable of bearing its own weight?

By Art. 10, page 343, a beam of oak a foot long and one inch square, whose weight is $\frac{1}{2}$lb. will just

just support 600 lbs; hence, in the expression $l = L\sqrt{\frac{W+2W'}{W}}$ (Art. 5), we have L = one foot, $W = \frac{1}{2}$ lb. and $W' = 600$, $\therefore l = \sqrt{\frac{\frac{1}{2}+1200}{\frac{1}{2}}} = \sqrt{2401}$ = 49 feet.

EXAM. II.

What must be the length of a bar of iron one inch square supported at one end, which would break by its own weight?

The length of a bar of iron of this thickness (supported at *both* ends) which would break by its own weight $\left(= L\sqrt{\frac{W+2W'}{W}}, \text{ where } L = \text{one foot}, W = 3lbs, W' = 2190 \right) = \sqrt{\frac{3+4380}{3}} = \sqrt{1461} = 38.2$ feet; the bar therefore which would just break by its own weight when supported at *one* end will be equal to *half* this length, or 19.1 feet. (Art. 7, page 353.)

EXAM. III.

Since a bar of iron an inch square and one foot long will support a weight of 2190 lbs, what must be the dimensions and weight of a similar bar which will break with its own weight, when supported at both ends?

By Art. 6, $l = \frac{L.\overline{W+2W'}}{W} = \frac{3+4380}{3}$ [a] $= 1461$ feet; and D (one inch or $\frac{1}{12}$ foot) : d :: L (1) : l (1461), $\therefore d = 121.75$; the *dimensions* therefore of the beam are

(a) For $L=1$, $W=3lbs$, $W'=2190lbs$.

are 1461 *feet long*, and the *side of its square* 121.75 feet; with respect to its *weight* (w), we have $w : W\,(3) :: l^3 : L^3 :: \overline{1491}^3 : 1$; consequently its weight is *immense*.

EXAM. IV.

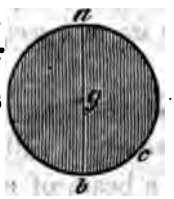
What must be the length of an oak prism 6 inches square, to break with its own weight when supported at both ends?

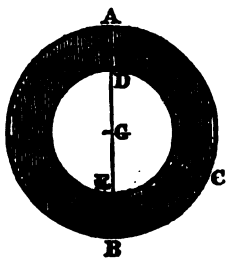
We must first find the length and weight of a beam six inches square which will just support some *given weight*. Now a beam of a *foot long* and an *inch square* just supports 600 *lbs;* if therefore L = the *length* of the beam *six inches* square which would just support 600 *lbs*, and D its *side*, we should have (if the weights of the beams be not taken into consideration) $S : s :: \frac{D^3}{L \times W'} : \frac{d^3}{l \times w'}$; and if $W' = w' = 600$, $S : s :: \frac{D^3}{L} : \frac{d^3}{l} :: \frac{6^3}{L} : \frac{1}{1}$; $\therefore$ when $S = s$, $\frac{6^3}{L} = 1$, or $L = 216$ feet = the *length* of a beam 6 inches square which would just support 600 *lbs.*; its weight (W) : weight of beam a foot long and one inch square ($\frac{1}{2}$ *lb.*) :: $L \times D^2 : l \times d^2 :: 216 \times 36 : 1 \times 1$, $\therefore$ $W = 216 \times 18 = 3888$ *lbs.*; hence the length of a beam six inches square which would break with its *own weight* $= L\sqrt{\frac{W + 2W'}{W}} = 216 \times \sqrt{\frac{3888 + 1200}{3888}} = 246.24$ feet.

XLV.

On the comparative strength of solid and hollow beams.

1. Let $ABCD, abcd$ represent the *lateral* sections of two cylindric beams of the same length and the same materials whose *weights* are W, w, one of which ($ABCD$) is *hollow*, and the other ($abcd$) is *solid*; and suppose the *annulus* $AEBFC$ to be *concentric* with the outer surface ABC of the cylinder, in which case the *center of gravity* of the annulus will be in the center of the *section* (G). Let S, s represent the strength of the *hollow* and *solid* beams respectively, then (by Art. 3, page 337, since $L = l$)





$$S : s :: \frac{\text{annulus } AEBFC \times AG}{W} : \frac{\text{area } abc \times ag}{w},$$

and if the *annulus* be equal to the *area* abc, then $S : s :: AG : ag$[a] $:: AB : ab$, i.e. the relative strengths of two cylindric beams of the *same length* and containing the *same quantity of matter*, one of which is *hollow* and the other *solid*, are as the *diameters of their respective lateral sections*.

2. From hence it appears that the strength of a cylindric

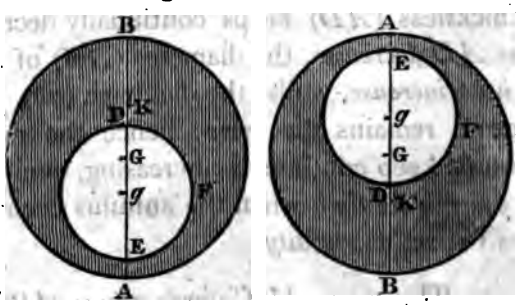
(a) For since the *lengths* of the beams are the same, when *annulus* $AEBFC$ = *area* abc, then $W = w$.

cylindric beam *consisting of a given quantity of matter* may be increased ad libitum by spreading that matter into the form of an *annulus* whose thickness (AD) keeps continually decreasing; for as AD *decreases*, the diameter (AB) of the annulus must *increase*, whilst the diameter (ab) of the *solid* beam remains the *same*; hence the ratio of $S : s$ would keep continually increasing, and would only come to its *limit* when the annulus becomes so thin as to lose its *rigidity*.

3. The area ABC : area $abc :: AB^2 : ab^2$, and the area DEF : area $abc :: DE^2 : ab^2$, $\therefore$ area ABC – area DEF (or the *annulus* $AEBFC$) : area $abc :: AB^2 - DE^2 : ab^2$; and when annulus $AEBFC$ = area abc, $AB^2 - DE^2 = ab^2$ or $ab = \sqrt{AB^2 - DE^2}$; hence the diameter of a *solid* cylinder which shall contain the same quantity of matter (in a given length), with a *hollow* one whose *exterior* diameter is AB and *interior* diameter DE, is equal to $\sqrt{AB^2 - DE^2}$; and the strength of the *hollow* beam : the strength of the *solid* one :: ($AB : ab ::$) $AB : \sqrt{AB^2 - DE^2}$.

4. If the hollow part of the beam and the annulus be not *concentric*, but have their diameters in the same straight line (AB), then the center of gravity of the annulus will not be in the center (G) of the circle ABC, but in some point (K) in the line AB, towards the *thicker* part of the annulus; and its distance AK from the point A will be *greatest* when the two circles ABC, DEF touch each other

other in the point A[a]; consequently, if the *fracture terminates* at that point, the strength of the beam will be the *greatest* when the thickness of the annulus at A is made as small as the nature of the substance will admit of[b].



Now in beams supported at *both* ends, the fracture terminates at the *upper*; and in beams supported only at *one* end, at the *lower* side of the beam; hence when the beam is supported at *both* ends, it will

(a) For let the area of the given circle $ABC=a$, of $DEF=b$, then the area of the *annulus* $=a-b$; let $AG=r$ (G being the center of the circle ABC), $Ag=x$ (g being the center of the circle DEF), and $AK=y$; then, by the property of the center of gravity, *the area of the circle* ABC × *the distance of its center of gravity from the point* A, must be equal to *the sum of the products of its two parts* (viz. *the annulus and the circle* DEF) *multiplied by the distance of their centers of gravity from the same point*, i.e. $ar=bx+\overline{a-b}.y$, $\therefore b\dot{x}+\overline{a-b}.\dot{y}=0$, or $\dot{x}=-\frac{\overline{a-b}.\dot{y}}{b}$; and as the fluxions of x and y have *different signs*, x must *decrease* when y *increases*, and *vice versa*; hence AK will be the *greatest* when Ag is the *least*, i.e. *when the circles touch each* other in the *point* A.

(b) For the strength of the beam varies as the *area of the annulus* × *the distance of its center of gravity from the point where the fracture terminates*; and as the area of the annulus is *given*, the strength will vary as AK.

will be *strongest* when the *thinner* part of the annulus is uppermost; and when it is supported only at *one* end, it will be strongest when the *thicker* part is in that position.

EXAM. I.

What weight could be sustained at the middle point of a cylindrical iron tube 8 feet long, whose diameter is $1\frac{1}{2}$ inches, and thickness $\frac{1}{4}$ of an inch; supposing the tube to be supported at both ends?

By Art. 3, the diameter of a *solid* cylinder of the same length and weight $= (\sqrt{AB^2 - DE^2}$, Fig. in page 356) $\sqrt{\frac{9}{4} - 1} = \sqrt{\frac{5}{4}} = 1.11$ inches; and the strength of this cylinder at its middle point : strength of a bar of iron *one foot long and one inch square* $:: \left(\frac{A \times G}{L \times W'} : \frac{a \times g}{l \times w}\right.$, where $w' = 2190$, the weight of the *bars themselves* not being taken into consideration) $\frac{.531}{8 \times W'} : \frac{.3927}{2190}$[c], $\therefore W' = 370.2$ *lbs.*; but by Art. 1, the strength of the *tube* : strength of the

(c) A=area of a circle whose diameter is $1.11 = \overline{1.11}^2 \times .7854 = .967$; $G = \frac{1.11}{2} = .55$, $\therefore A \times G = .967 \times .55 = .531$; $L = 8$, $\therefore \frac{A \times G}{L \times W'} = \frac{.531}{8W'}$. Also a = (area of a circle whose diameter is 1) $.7854$, $g = \frac{1}{2}$, $l = 1$, $\therefore \frac{a \times g}{l \times w'} = \frac{.7854 \times \frac{1}{2}}{2190} = \frac{.3927}{2190}$.

the cylinder :: *diameter of the tube* : *diameter of the cylinder* :: $\frac{3}{2}$: 1.11 :: 3 : 2.22, ∴ the *tube* would sustain at its *middle point* a weight of $\frac{3 \times 370.2}{2.22}$ or about 502 *lbs.*

EXAM. II.

Compare the lateral strengths of two cylindrical iron tubes of the same length, of the same diameter (viz. 4 inches), and of the same bore (viz. 3 inches); in the former of which the bore is brought within $\frac{1}{8}$th of an inch of the upper surface, and in the latter it is concentric with the tube itself; taking the case when the tubes are supported at both ends.

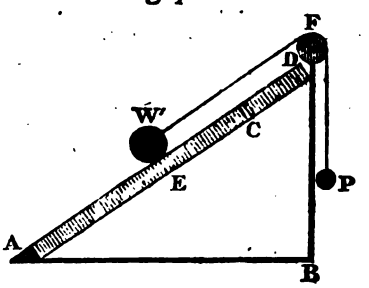
Since the *lengths, weights,* and *areas of the annuli* of these tubes are *given,* their lateral strengths will be as the *distance of the centers of gravity of the annuli from the upper surface.* In the tube whose bore is *concentric with the tube itself,* the center of gravity of the annulus will be in the *axis of the cylinder,* and consequently its *depth* will be *two inches*: the other comes under the case of Art. 4, (See Fig. in page 358) where $AB = 4$ inches, $DE = 3$ inches, $AG = 2$ inches, and $AE = \frac{1}{8}$th of an inch, and therefore $Ag = 1\frac{7}{8} = \frac{15}{8}$ inches. Let $p = .7854$, then the area of the circle $ABC = p \times 4^2 = 16p$; area of circle $DEF = p \times 3^2 = 9p$, ∴ area of the *annulus* $= 16p - 9p = 7p$; substitute these values for a, b, and $a - b$ in the equation $ar = bx + \overline{a-b}.y$ of Note *a*, page 358, and we have $16pr = 9px + 7py$.

$+7py$, or $y\,(AK) = \frac{16r-9x}{7} = \left(\text{for } r=2, x=\frac{13}{8}\right)$ $\frac{139}{56}$; hence the strength of the *former* tube : that of the *latter* (:: $AK : AG$) :: $\frac{139}{56} : 2 :: 139 : 112$.

XLVI.

On the strength and stress of beams placed obliquely.

1. By Art. 3, page 274, if a body be sustained upon an inclined plane by a force acting *parallel to the plane*, the weight : the *pressure* upon the plane :: *length* of the plane : *base* of the plane; if therefore a weight W' be supported at the *middle point* of a rectangular beam AD, in the manner represented in the annexed Figure, then W' : *stress* upon the beam :: $AD : AB$:: *radius* : *cosine of* $\angle BAD$; and as the *weight of the beam* may also be considered as a weight acting at the *middle point*, the stress arising from this cause will also be diminished in the same ratio; i. e. if W = *weight of the beam*, then W : stress arising from its own weight :: *radius* : *cos.* $\angle BAD$; hence $W+W'$: *whole stress arising from* $W+W'$:: radius : cos. BAD or cosine of the $\angle$ of *elevation* of the beam above the horizon; let $\angle$ of elevation = α and radius = 1, then



then $W + W'$: stress upon the inclined beam AD :: 1 : cos. α, $\therefore$ the *stress* $= \overline{W + W'}$. cos. α.

2. Hence in beams of *different lengths* the *stress* will vary as $L \times \overline{W + W'} \times$ cos. α. But when a *rectangular* beam is inclined to the horizon, the *area of a vertical section* is increased in the ratio of rad. : cos. α; and the *depth of the center of gravity* of this area is *also* increased in the same ratio; $\therefore$ the *tendency to resist fracture* is increased in the ratio of $\overline{\text{rad.}^2}$, (1)^2 : $\overline{\text{cos. } \alpha}^2$; let then S, s represent the *strengths*; B, b the *breadths*; D, d the *depths*; W, w the *weights*; W', w' the *weights supported at their middle points*; L, l the *lengths*; and α, α' the *angles of elevation*; of two *rectangular* beams, and we shall have $S : s :: \dfrac{B \times D^2 \times \dfrac{1}{\overline{\text{cos. } \alpha}^2}}{L . \overline{W + W'} . \text{cos. } \alpha}$

$: \dfrac{b \times d^2 \times \dfrac{1}{\overline{\text{cos. } \alpha'}^2}}{l . \overline{w + w'} \text{ cos. } \alpha'} :: \dfrac{BD^2}{L . \overline{W + W'} . \overline{\text{cos. } \alpha}^3} : \dfrac{bd^2}{l . \overline{w + w'} . \overline{\text{cos. } \alpha'}^3}$;

and the strength of a *given beam* to support a given weight will vary *inversely as the cube of the cosine of its elevation.*

3. This expresses the relation of $S : s$ at the *middle* point of beams inclined to the horizon in the $\angle^s \alpha, \alpha'$; but since the pressure arising from the weight is diminished in the same ratio at *every point* of a plane inclined to the horizon in a given angle (when the weight is sustained upon the plane by

by a power acting *parallel to* the plane), the strength at any other point C may be found by applying the proportion of Art. 8, page 341, to the value of S thus found, viz. S (or strength at the *middle point* E of the beam) : strength at the point $C :: AC \times CD : \frac{1}{4}AD^2$.

EXAM. I.

What is the greatest weight which may be rolled down an oak plank 20 feet long, 18 inches broad, 3 inches thick, and inclined to the horizon in an ∠ of 60°, without breaking it; the weight of the plank not being taken into the consideration?

To find the weight which this plank would sustain at its *middle* point when placed *horizontally*, we have $S : s :: \frac{B \times D^2}{L \times W'} : \frac{b \times d^2}{l \times w}$ [a] $:: \frac{18 \times 9}{20 \times W'} : \frac{1 \times 1^2}{1 \times 600}$, $\therefore W' = 4860\,lbs$; but the strength of a plane *inclined to the horizon in an ∠ of* 60° : its strength when placed *horizontally* :: $\overline{\text{rad.}}^3 : \overline{\text{cos. } 60°}^3$, $\therefore$ the *inclined* plank would sustain $8W'$ or $38880\,lbs$ = about $17\frac{5}{14}$ tons at its middle point; consequently as the stress at this point is the *greatest*, a weight of $17\frac{5}{14}$ tons might be rolled down without breaking it.

EXAM. II.

Two iron bars, one of which is ten feet long and 3 inches square, the other 4 feet long and one inch square, are supported at both ends; the former is inclined

(a) Here b, d, l, &c. represent the same quantities as in Exam. 2. page 345.

inclined to the horizon in an $\angle$ *of* 60°, *and supports a weight of* 3000 *lbs*; *the latter is inclined to the horizon in an* $\angle$ *of* 30°, *and supports a weight of* 500 *lbs*; *Which is most likely to break?*

Here $S : s :: \left(\frac{D^3}{L.W+W'.\cos.a}\right)^3 : \frac{d^3}{l.w+w'.\cos.a'}\Big|^3 ::$ (a)

$\frac{27}{10.270+3000.\cos.60°}\Big|^3 : \frac{1}{4.12+500.\cos.30°}\Big|^3$ (a)

$:: \frac{216}{32700} : \frac{1}{768\sqrt{3}} :: 8.77 : 1$, $\therefore S$ is greater than s, and consequently the *ten-foot bar* is least likely to break.

XLVII.

On the pressure and thrust of beams placed obliquely.

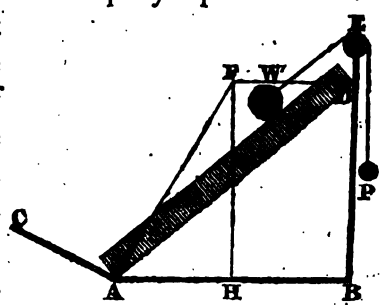
We shall conclude this Lecture with a few observations upon the *pressure and thrust* of beams placed obliquely.

1. In Exam. 5, page 281, it was shewn that if a body be supported between two inclined planes, and 2 lines HF, LF be drawn from the points of contact at right angles to the plane meeting in a point (F), and from that point another line (FG) be drawn at right angles to the horizon, then the 3 lines FH, FG, FL will be in the direction of the 3 forces which keep the body at rest. Now since FG represents the direction in which the weight of the

(a) Since the weight of a bar of iron *one foot long* and an *inch square* is 3 *lbs*, the weight of one *four* feet long and an inch square will be 12 *lbs*, $\therefore w=12$, and $W : w$ (12) $:: L \times D^2 : l \times d^2$ $:: 10 \times 9 : 4 \times 1$. $\therefore W=270$ *lbs*.

the body acts, it must pass through its *center of gravity;* the point F might therefore be determined by drawing LF at right angles to the plane AC, and another line GF passing through the center of gravity at right angles to the horizon; and if FH be joined, and DC drawn at right angles to FH, it would give the position of the other plane (Fig. page 281).

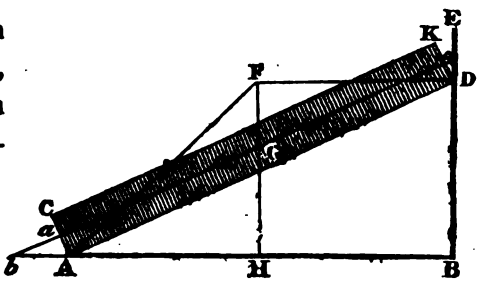
2. Suppose now the beam AD laden with the weight W' to lean obliquely upon the wall EB; let the weight be so placed upon the beam that its center of gravity may lie in the same vertical plane with the center of gravity of the beam, and let G be the *common center of gravity of the weight and beam.* Draw DF at right angles to EB, and through G draw FH at right angles to the horizon; join FA, and draw AC at right angles to FA; then AC will be the position of a plane which will just prevent the beam from sliding, and consequently keep it at rest in the given position AD. The three forces therefore which keep the beam at rest, are, 1. The *re-action of the wall* in direction DF or HA; 2. Its *own weight and that of* W' in direction FH; and 3. The *re-action of the plane* AC in direction AF; these three forces may consequently be represented



by

by the three sides of the triangle FAH; and in estimating the effect produced by the pressure of the beam, we may say that AH represents the *pressure upon the wall* at D; and FA the pressure or *thrust* at A.

3. Suppose the beam to be *prismatic*, and to press only with *its own weight*; then draw the *axis* ac, and bisect it in G; G is the *center of gravity*. Produce ca and BA till they meet in b; then



if the *depth* of the beam be inconsiderable when compared with its *length*, and its inclination to the horizon be not less than about 30°, the sides Ab, ba of the triangle Aba will bear a very small ratio to aG, AH, and may therefore be neglected in the following computation without producing any material error [a].

Let

(a) For let DAB or $abA = 30°$, then $Ab = 2Aa =$ depth of the beam (d); and $ab = Ab \times \cos. 60° = \frac{d\sqrt{3}}{2}$. Suppose the beam to be 20 feet long and 3 inches deep, then $Ab = 3$ inches, and $ab = \frac{3\sqrt{3}}{2} = 2.59$ inches; $aG = 10$ feet $= 120$ inches; $AH =$ (cos. 60° to radius aG *nearly* $=$) $\frac{120\sqrt{3}}{2} = 103.92$ inches; $\therefore ab : aG :: 2.59 : 120$ or $ab = \frac{aG \times 2.59}{120} =$ about $\frac{1}{46}$th aG; and $Ab : AH :: 3 : 103.92$, $\therefore Ab = \frac{AH \times 3}{103.92} =$ about $\frac{1}{34}$th AH. As the

Let L = the *length* of the beam, α = its *angle of inclination* to the horizon, and W = *weight* of the beam; then $AD\ (L) : DB$ or $FH ::$ radius $(1) :$ sin. α, $\therefore FH = L \times$ sin. α; also $bG\ (= \alpha G = \frac{1}{2}L) : bH$ (or AH) :: radius (1) : cos. α, $\therefore AH = \frac{1}{2}L \times$ cos. α; hence $AF = \sqrt{FH^2 + AH^2} = L \times \sqrt{\text{sin. } \alpha|^2 + \frac{1}{4}\text{cos. } \alpha|^2}$. Now the *pressure at D* (P) : *weight* $(W) :: AH : FH :: \frac{1}{2}L \times$ cos. $\alpha : L \times$ sin. α, $\therefore P = \frac{W \times \text{cos. } \alpha}{2 \text{ sin. } \alpha} = \frac{W}{2 \text{ tan. } \alpha}$; and the *thrust at A* (T) : the *weight* $(W) :: FA : FH :: L . \sqrt{\text{sin. } \alpha|^2 + \frac{1}{4}\text{cos. } \alpha|^2} : L \times$ sin. α, $\therefore T = \frac{W\sqrt{\text{sin. } \alpha|^2 + \frac{1}{4}\text{cos. } \alpha|^2}}{\text{sin. } \alpha}$.

4. Since $P = \frac{W}{2 \text{ tan. } \alpha}$; if tan. $\alpha = \frac{1}{2}$, then $P = W$; if $\alpha = 90°$, then tan. α is infinite, and $P = 0$. Now the angle whose tangent $= \frac{1}{2}$ is 26°. 34′; hence, when the beam is inclined to the wall in angle of 26°. 34′, the *pressure* against the wall is equal to its *weight*; and whilst the angle of inclination increases from 26°. 34′ to 90°, this pressure varies through all degrees of magnitude from W to 0.

Again,

the angle of inclination *increases*, the triangle Aba keeps continually diminishing, therefore the error arising from this mode of computation will *also* be diminished; but as the inclination *decreases*, the sides Ab, ba of the triangle Aba continually *increase*, $\therefore$ the error would be very great when the angle of inclination is *small*.

Again, when $\tan.\alpha = \frac{1}{2}$, $\frac{\sin.\alpha}{\cos.\alpha} = \frac{1}{2}$, $\therefore \cos.\alpha = 2\sin.\alpha$; hence $T = \left(\frac{W\sqrt{\sin.\alpha^2 + \frac{1}{4}\cos.\alpha^2}}{\sin.\alpha} = \right)\frac{W\sqrt{2\sin.\alpha^2}}{\sin.\alpha}$ $= W\sqrt{2}$; let $\alpha = 90°$, then $\sin.\alpha = 1$, and $\cos.\alpha = 0$, $\therefore T = W$. Whilst the inclination of the beam therefore increases from 26°. 34′ to 90°, the *thrust* at A varies through all degrees of magnitude from $W\sqrt{2}$ to W.[a]

5. This theorem would evidently apply to a beam *laden with a given weight* (W') *at its middle point*; for in this case FH would pass through the center of gravity both of the weight and beam, and consequently might represent the whole weight $W + W'$; the *pressure against the wall* (P) of a beam thus laden would therefore be equal to $\frac{W + W'}{2\tan.\alpha}$, and the *thrust at the bottom* (T) to $\frac{W + W'.\sqrt{\sin.\alpha^2 + \frac{1}{4}\cos.\alpha^2}}{\sin.\alpha}$.

" Suppose, for instance, a ladder of uniform thick-
" ness, whose weight is 50 pounds, to be placed
" against a wall at an inclination of 60° to the
" horizon; that a person weighing 12 stone ascends
" up

(a) We have here applied our calculation to an angle somewhat less than 30°; the *error* therefore arising from it will be somewhat greater than what is estimated in the preceding Note; but at the same time it may be observed, that if they be *very* long and *very* thin, this mode of computation will approximate very nearly to the truth for beams inclined at *much smaller angles*.

" up it with a burden of 100*lbs*; and that it is re-
" quired to ascertain the pressure against the wall
" and the thrust at the bottom, when he arrives at
" the *middle stave*; in this case $W = 50$, $W' = 268$,
" $\alpha = 60°$, $\therefore P = \frac{318}{2\sqrt{3}}$ = about 90*lbs*.; and $T =$
" $\frac{318 \times \sqrt{13} \times 2}{4\sqrt{3}}$ = about 330*lbs*."

Some very excellent practical reflections upon the subject of several parts of this Lecture will be found in Section 12, 13, 14, 15, 16, of Chap. 3, Book II, of MACLAURIN'S *Account of Sir Isaac Newton's Philosophical Discoveries*.

END OF PART II.

VOL. II.

AN

INTRODUCTION

TO THE

STUDY OF NATURAL PHILOSOPHY.

PART III.

ON THE

MOTION OF BODIES UPON INCLINED PLANES;

AND ON THE

ROTATORY AND VIBRATORY MOTION OF BODIES.

Lecture XII.

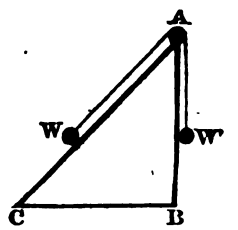
ON THE MOTION OF BODIES UPON INCLINED PLANES.

IN the Second Lecture, we investigated the relation which takes place between the *space* described, the *velocity* acquired, and the *time* of its motion, when a body ascends or descends *perpendicularly* near the Earth's surface. The object of this Lecture is, to ascertain that relation when the bodies ascend or descend upon planes *inclined to the horizon.*

XLVIII.

On the mode of ascertaining the space *described, the* velocity *acquired, and the* time *of its motion, when a body ascends or descends upon an inclined plane.*

I. In Lecture 9, Sect. xxxvi, it was shewn, that if a weight (W) be sustained upon an inclined plane AC by another weight (W') acting upon it in a direction *parallel to the plane*, then $W' : W :: AB : AC$. Suppose now the string WAW' to be cut in two, then it is evident that the weight W would descend down the plane with a *force* which bears to its *own weight* the ratio of $AB : AC$; and since (by Art. 8. page 276, Part I.) this force is *constant* through every part of the plane, the body thus descending may be considered as acted upon in every point of its descent by a *constant force*, which bears to the *force of gravity* (see page 21, Part I.) the given ratio of the *height of the plane* (H) to the *length of the plane* (L); call this force F, and let the force of gravity be represented by *unity*, then $F : 1 :: H : L$, and $F = \frac{H}{L}$[a]; "i. e. the force which accelerates the "motion



(a) Since $F : 1 :: AB : AC ::$ sin. of $\angle ACB$: rad. $\therefore F = \frac{\text{sin. of } \angle ACB}{\text{rad.}} \propto$ sin. $\angle ACB$ (for rad. is constant); the force which accelerates a body down an inclined plane, therefore, varies as the *sine of the angle of the plane's elevation.*

" motion of a body down an inclined plane is such " a part of the force of gravity, as may be repre- " sented by the *fraction* $\frac{H}{L}$; this force therefore dif- " fers not from the force of gravity in *kind*, but in " *degree*; the effects produced by it must conse- " quently be analogous to the effects produced by " *gravity*."

2. In order to estimate these effects, we have only to consider, that if a body be acted upon by *different* constant forces for the *same time*, the *velocity generated* will evidently be proportional to the *intensity* of those forces; and that if it be acted upon by the *same* force for *different times*, the velocity will be proportional to the *time* for which the forces act; from which it follows, that if a body be acted upon by *different* constant forces for *different* times, the *whole velocity generated* will be as the *force and time conjointly*. Suppose now that the force of gravity is represented by *unity*, that $m = 16\frac{1}{12}$ feet, and that V = the *velocity* acquired in the *time* T whilst a body describes the *space* S acted upon by *some other constant force* F, then, from what has just been shewn, V: the velocity acquired by *gravity* in $1''$ ($2m$[b]) :: $F \times T : 1 \times 1$, $\therefore V = 2mFT$, and $T = \frac{V}{2mF}$. Again, since $V \propto F \times T$, $T \times V$ must vary

([b]) Art. 1, page 33, Part I.

vary as $F \times T^2$; but $S \propto T \times V$[a], $\therefore S \propto F \times T^2$; hence S : the space described by *gravity* in 1″ (m) :: $F \times T^2 : 1 \times 1^2$, $\therefore S = mFT^2$, and $T = \sqrt{\frac{S}{mF}}$. Lastly, since $V \propto F \times T$, V^2 must vary as $F \times T \times V$, or as $F \times S$, $\therefore V^2$: square of the velocity acquired by gravity in 1″ $(4m^2)$:: $F \times S : 1 \times$ space described by gravity in 1″ :: $F \times S : 1 \times m$; hence $V^2 = 4mFS$, $V = 2\sqrt{mFS}$, and $S = \frac{V^2}{4mF}$.

3. Let us now apply these expressions to the case before us, *i. e.* let a body descend down an inclined plane whose *height* is (H) and *length* (L); then $F = \frac{H}{L}$; and if the body descends from *rest*,

Space (S) described in the *time* $T = \frac{H}{L} \times mT^2$, and $T = \sqrt{\frac{L \times S}{mH}}$.

Velocity (V) acquired in time $T = \frac{H}{L} \times 2mT$, and $T = \frac{L \times V}{2mH}$.

V^2 acquired thro' *space* $(S) = 2\sqrt{\frac{H}{L} \times mS}$, and $S = \frac{L \times V^2}{4mH}$.

4. Since

(a) The space described by a body acted upon by a *constant force* (i. e. by a force which generates equal increments of velocity in equal times) will always be represented by a *right-angled triangle*, one of whose sides represents the *time*, the other the *velocity acquired*; consequently the *space* will always be proportional to the *time and velocity conjointly*.

4. Since $S = \frac{H}{L} \times m T^2$, and H, L, m are *given*, the space described varies as the square of the time when a body falls from rest down an *inclined plane*, as well as when it descends freely by the *force of gravity*; the spaces described from rest in *equal successive portions of time* will therefore be as the odd numbers 1, 3, 5, 7, &c.; and if the body be projected *upwards* with the velocity acquired in falling through any space upon the plane, it will ascend to the point from which it fell, the spaces described in *equal successive portions of time* being as the numbers 1, 3, 5, 7, &c. taken in the *inverted* order. If, moreover, at any point of its descent, it moves forward with the velocity acquired *continued uniformly*, it will describe *twice* the space in the same time as that in which it has fallen to acquire the velocity; and if it be projected *downwards* or *upwards* with *velocity* (*V*), and moves for the *time* (*T*), the *space* described in that time will be equal to $T \times V \pm \frac{H}{L} \times m T^2$. All this follows from the *law of acceleration and retardation* of bodies moving upon *inclined planes* being the same as that which regulates the motion of bodies descending or ascending freely by the force of gravity. (See Art. 7..11, pages 27..32, Part I.

5. The expressions contained in Art. 3. apply to the case of a body descending from rest through any *part* of an inclined plane whose height is (*H*) and

and length (L). If the body falls through the *whole length*, then $S=L$, $\therefore$ the velocity acquired in falling down the whole length of the plane $\left(=2\sqrt{\frac{H}{L}\times mS}\right)=2\sqrt{mH}=$ (by Art. 1, page 33, Part I.) *the velocity acquired by descending freely through the height* H; $T\left(=\sqrt{\frac{L\times S}{mH}}\right)=\frac{L}{\sqrt{mH}}$; but the time of falling *freely* down $H=\sqrt{\frac{H}{m}}$, $\therefore$ the time of describing the *whole length* of the plane : the time of falling *freely down its height* :: $\frac{L}{\sqrt{mH}}$: $\sqrt{\frac{H}{m}}$:: L : H :: *length* of the plane : *height* of the plane.

6. Since $T=\frac{L}{\sqrt{mH}}$, $V=\sqrt{mH}$, and m is a given quantity, T *varies* as $\frac{L}{\sqrt{H}}$, and V as $\sqrt{H}$; i. e. the *time* of describing any inclined plane varies as its length *directly*, and the square root of its height *inversely*; and the *velocity acquired* varies as the *square root of the height*, whatever be the *length* of the plane[a].

Ex.

([a]) The expressions deduced in this Section are only true when the body *slides* down a *perfectly smooth* plane; for in this case it is evident that every particle of the body is *equally accelerated*, and therefore whatever is proved of any one point of it will apply equally

EXAM. I.

How far will a body descend from rest in 4″, *upon an inclined plane whose length is* 400 *feet and height* 300 *feet?*

Here $L=400$, $H=300$, $T=4''$ $\Big\}$ $\therefore S=\frac{H}{L}\times mT^2=\frac{300}{400}\times 16\frac{1}{12}\times 16$ $=193$ feet.

EXAM. II.

How long would a body be in falling down 100 *feet of a plane, whose length is* 150 *feet and height* 60 *feet?*

Here $L=150$, $H=60$, $S=100$ $\Big\}$ $\therefore T=\sqrt{\frac{L\times S}{mH}}=\sqrt{\frac{150\times 100}{16\frac{1}{12}\times 60}}$ $=3.9$ seconds.

EXAM. III.

The length of an inclined plane is 60 *feet, and its elevation* 30°; *What velocity would a body acquire in falling from rest down it for* 2″.

Here

$L=60$, $H=\sin.\ 30^\circ$ to rad. $L=30$, $T=2$ $\Big\}$ $\therefore \quad V=\frac{H}{L}\times 2mT=\frac{1}{2}$ $\times 32\frac{1}{6}\times 2=32\frac{1}{6}$ feet in 1″.

Ex.

equally to all; but if the body in its fall has a *rotatory* motion communicated to it, then it is evident that all the points of it will *not* be equally accelerated, and the force of acceleration of its *center of gravity* must be estimated according to the principles hereafter to be explained in the 14th Lecture.

EXAM. IV.

The height of plane : length of a plane :: 7 : 15; how long would a body be in falling down it, to acquire a velocity of 20 feet per second?

Here $\frac{H}{L}=\frac{7}{15}$, $\therefore T=\frac{L\times V}{2mH}=\frac{15\times 20}{32\frac{1}{2}\times 7}=1.3$ seconds.

EXAM. V.

$H : L$:: 5 : 14; What space must a body fall through, to acquire a velocity of 10 feet per second?

Here $\frac{H}{L}=\frac{5}{14}$, $\therefore S=\frac{L\times V^2}{4mH}=\frac{14\times 100}{64\frac{1}{2}\times 5}=4.3$ feet.

EXAM. VI.

$H : L$:: 25 : 90; What velocity would a body acquire in falling down 70 feet?

$$V=\sqrt{\frac{H\times 4mS}{L}}=\sqrt{\frac{25\times 64\frac{1}{2}\times 70}{90}}=35.3 \text{ feet in } 1''.$$

EXAM. VII.

The length of an inclined plane is 100 feet, and its elevation 60°; how long would a body be in falling down it, and what velocity would it acquire at the end of its fall?

Here L (100) : H :: rad. : sin. 60° :: 1 : $\frac{\sqrt{3}}{2}$,

$\therefore H=\frac{100\sqrt{3}}{2}=50\sqrt{3}$; by Art. 5, $T=\frac{L}{\sqrt{mH}}=$

100

$\sqrt{\frac{100}{16\frac{1}{12}\times 50\sqrt{3}}} = 2.6$ seconds; also $V = 2\sqrt{mH} =$

$= 2\sqrt{16\frac{1}{12}\times 50\sqrt{3}} = 74.4$ feet in 1″.

EXAM. VIII.

A body is projected up an inclined plane, whose length is 10 *times its height, with a velocity of* 30 *feet in* 1″; *in what time will its velocity be destroyed?*

The *time* in which a body would fall down an inclined plane of this elevation to acquire a velocity of 30 feet per second $= \left(\frac{L\times V}{2mH} =\right) \frac{10\times 30}{32\frac{1}{6}\times 1} = 9.3''$.

EXAM. IX.

A body is projected up an inclined plane, whose height is $\frac{1}{6}$*th of its length, with a velocity of* 50 *ft. per second; Find its place, and velocity after* 6″ *are elapsed.*

This is a case of Art. 4, where $V = 50$, $T = 6$, $\frac{H}{L} = \frac{1}{6}$, $\therefore$ the space described in 6″ $\left(= T\times V - \frac{H}{L}\times mT^2\right) = 6\times 50 - \frac{1}{6}\times 16\frac{1}{12}\times 36 = 300 - 96\frac{1}{2} =$ $203\frac{1}{2}$ feet from the bottom of the plane.

The *velocity* generated or destroyed upon a plane of this elevation in 6″ $\left(= \frac{H}{L}\times 2mT\right) = \frac{1}{6}\times 32\frac{1}{6}$ $\times 6 = 32\frac{1}{6}$ feet in 1″; hence the velocity of the

body after it has ascended up the plane for 6″ will be $50-32\frac{1}{6}$ or $17\frac{5}{6}$ feet in 1″.

EXAM. X.

Two bodies, projected along two planes inclined to the horizon in ∠ˢ of 45° and 30°, describe spaces respectively as $\sqrt{2} : \sqrt{3}$ *before all their velocity is destroyed; Required the ratio of the velocities with which the bodies were projected up the planes.*

Let S, s represent the *spaces* described by the bodies on the planes; V, v the *velocities* with which they are projected up the planes; F, f the *accelerative forces* down the two planes. Now the velocities with which the bodies are *projected upwards* will evidently be such as they would respectively acquire in *falling down* spaces which are to each other as $\sqrt{2} : \sqrt{3}$ with the accelerative forces down planes whose elevations are 45° and 30°; and by Art. 2, $V^2 \propto F \times S$;

Now by Note *a*, page 2,

$$F : f :: \sin. 45° : \sin. 30° :: \frac{1}{\sqrt{2}} : \frac{1}{2};$$

By the hypothesis,

$$S : s :: \ldots\ldots :: \sqrt{2} : \sqrt{3};$$

$$\therefore F \times S : f \times s :: 1 : \tfrac{1}{2}\sqrt{3} :: 2 : \sqrt{3};$$

$$\text{Hence } V^2 : v^2 \ldots :: 2 : \sqrt{3},$$

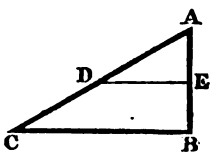
$$\text{and } V : v \ldots :: \sqrt{2} : \sqrt[4]{3}.$$

Ex.

EXAM. XI.

A body falls from rest down the inclined plane AC; Compare the times of describing the first *and* last *halves of it.*

Bisect AC in D, and draw DE parallel to CB; by Art. 6, the *time down* AC : the



time down $AD :: \frac{AC}{\sqrt{AB}} : \frac{AD}{\sqrt{AE}}$

:: (by sim. Δ^s) $\frac{AC}{\sqrt{AC}} : \frac{AD}{\sqrt{AD}}$

$:: \sqrt{AC} : \sqrt{AD}$ [a] $:: \sqrt{2} : 1$; hence the time down AC – the time down AD (i. e. *the time down* DC) : the *time down* $AD :: \sqrt{2} - 1 : 1$.

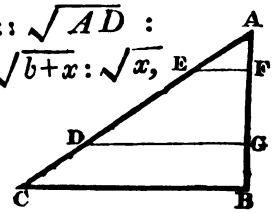
EXAM. XII.

To mark out upon the plane AC a part ED which shall be equal to the height AB, and which a body (falling down AC) would describe in the same time as one falling freely through AB.

Let $AC = a$
AB or $ED = b$
$AE = x$,
then $AD = b + x$

By Note to last Exam. [b]

$T^{AD} : T^{AE} :: \sqrt{AD} : \sqrt{AE} :: \sqrt{b+x} : \sqrt{x}$,



therefore

(a) From this it appears that the times down *different parts* of the *same* inclined plane (when the body falls from rest from the *top* of the plane) are to each other as the *square roots of the lengths of those parts.*

(b) T^{AD} means the *time* down AD, and so of the rest.

$\therefore T^{AD}-T^{AE}$ (i. e. T^{DE}) : T^{AE} :: $\sqrt{b+x}-\sqrt{x}$: $\sqrt{x}$,
but (Note to last Exam.) T^{AE} : T^{AC} :: $\sqrt{AE}$: $\sqrt{AC}$:: $\sqrt{x}$: $\sqrt{a}$,
and by Art. 5, T^{AC} : T^{AB} :: a : b :: $\sqrt{a}$: $\frac{b}{\sqrt{a}}$;

$\therefore$ ex æquo, T^{DE} : T^{AB} :: $\sqrt{b+x}-\sqrt{x}$: $\frac{b}{\sqrt{a}}$.

But $T^{DE}=T^{AB}$, $\therefore \sqrt{b+x}-\sqrt{x}=\frac{b}{\sqrt{a}}$;

which equation solved gives $x=\frac{\overline{a-b}|^2}{4a}$ or $AE=\frac{\overline{AC-AB}|^2}{4AC}$.

XLIX.

QUESTIONS FOR PRACTICE.

1. The *length* of an inclined plane is 400 feet, and the *height* 250; a body falls from rest from the top of the plane; What *space* (S) will it have fallen through in 4″; what *time* (T) will it be in falling through 200 feet; and what *velocity* (V) will it have acquired, when it has arrived within 40 feet of the bottom of the plane?

Answer, $S = 160.8$ feet; $T=4.4$ seconds; $V=60.1$ feet in 1″.

2. A body has been falling for 5″ down an inclined plane whose length is 3 times its height; What *velocity* (V) will it have acquired at the end of its fall?

Ans. $V=53\frac{11}{18}$ feet in 1″.

3. The *elevation* of a plane is 30°; a body in falling from the top to the bottom of it, acquires a *velocity* of 500 feet in 1″; What is the *length* (L) of the plane?

Ans. $L=7776$ feet.

4. Two

4. Two inclined planes have a *common height* of 50 feet; the elevation of one of them is 60°, of the other 30°; With what *velocity* (V) must a body be projected from the *bottom* of the former, that it may just rise to the *top* of the latter? and what will be the whole *time* (T) of its ascending and descending through the two planes?(a)

Ans. $V = 56.6$ feet in $1''$; $T = 5.15$ seconds.

5. How long will a body be in falling down the last half of a plane, whose *height* is 30 feet, and *elevation* 45°? Ans. .563 seconds.

6. The *length* of a plane is 150 feet, and *elevation* 30°; mark out upon it *a part equal to the height* which a body in falling down it describes, whilst another body would descend freely through the height.

Ans. *It begins to describe this part, when it has fallen through* 9.37 *feet from the top of the plane.*

L.

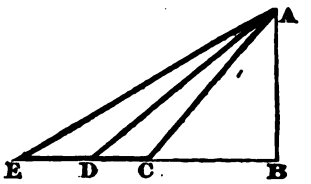
On the motion of bodies down different systems of Inclined Planes.

It has already been shewn, that when a body descends down an inclined plane whose *length* is L and *height* H, the *velocity* acquired varies as $\sqrt{H}$, and

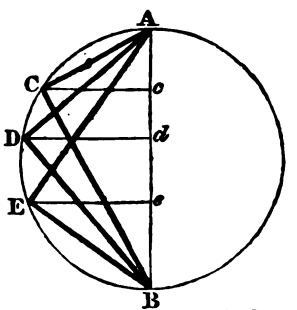
(a) The planes are placed as in Fig. p. 280, Part II.; a body is projected from D with a velocity just sufficient to carry it to A, and then falls from *rest* down the plane AC.

and the *time of description* as $\frac{L}{\sqrt{H}}$; let us now apply these expressions to finding the relation between the time and velocity of bodies falling down *different systems* of inclined planes.

1. Let AC, AD, AE, &c. be a system of inclined planes having the *same height* AB; then since the *velocities* acquired by bodies falling down these planes are as $\sqrt{AB}$, and the *times* of description as $\frac{AC}{\sqrt{AB}}$, $\frac{AD}{\sqrt{AB}}$, $\frac{AE}{\sqrt{AB}}$, &c. (i. e, as AC, AD, AE, &c.), it is evident that bodies falling down a system of planes of this kind would acquire at the end of their fall the *same velocity*[a], and that the *times* of description would be as their respective *lengths*.



2. Let the diameter AB of the circle ACB be *perpendicular to the horizon*; draw the chords AC, AD, AE, &c.; CB, DB, EB, &c.; draw also Cc, Dd, Ee, &c. *parallel to the horizon*; then the *velocities* acquired by bodies falling down the *former* system of chords are as $\sqrt{Ac}$, $\sqrt{Ad}$, $\sqrt{Ae}$, &c. and down the *latter* as $\sqrt{cB}$,



(a) And by Art. 5. page 5, this velocity is equal to the velocity which a body would acquire by falling freely through the height (AB) of the plane.

$\sqrt{cB}$, $\sqrt{dB}$, $\sqrt{eB}$, &c. but $Ac=\frac{AC^2}{AB}$[b], $Ad=\frac{AD^2}{AB}$, &c.; and $cB=\frac{CB^2}{AB}$, $dB=\frac{DB^2}{AB}$, &c.; hence $\sqrt{Ac}$, $\sqrt{Ad}$, $\sqrt{Ae}$, &c. vary as AC, AD, AE, &c. and $\sqrt{cB}$, $\sqrt{dB}$, $\sqrt{eB}$, &c. vary as CB, DB, EB, &c. (AB being *constant*); the *velocities* therefore acquired by bodies falling down these two different systems of chords are respectively as the *lengths of the chords* in each system [c].

3. The times of describing AC, AD, AE, &c. and CB, DB, EB, &c. are as $\frac{AC}{\sqrt{Ac}}$, $\frac{AD}{\sqrt{Ad}}$, $\frac{AE}{\sqrt{Ae}}$, &c.; and $\frac{CB}{\sqrt{cB}}$, $\frac{DB}{\sqrt{dB}}$, $\frac{EB}{\sqrt{eB}}$, &c.; but each of these quantities is equal to $\sqrt{AB}$[d]; the *times* therefore

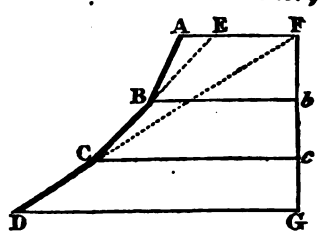
(b) For (Euc. 8. 6.) $Ac : AC :: AC : AB$, or $Ac=\frac{AC^2}{AB}$; and so of the rest.

(c) Hence the velocity acquired in falling down any one of these chords (AC for instance) : velocity acquired in falling down the *diameter* :: $AC : AB$. To find therefore the *length of a chord* (AC) down which a body ought to fall to acquire $\frac{1}{n}$th part of the velocity which it would acquire in falling down the *diameter*, we have only to take $AC=\frac{1}{n}AB$; thus if it be required to find the chord down which a body must fall to acquire *half* the velocity down AB, then $AC=\frac{1}{2}AB$=radius=chord 60°; &c. &c. &c.

(d) For since $Ac : AC :: AC : AB$, $\frac{AC^2}{Ac}=AB$, $\therefore \frac{AC}{\sqrt{Ac}}=\sqrt{AB}$; and so of the rest.

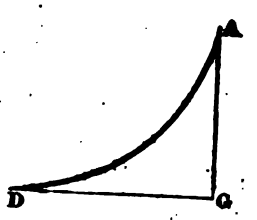
therefore of falling down the *chords* of a circle whose diameter is placed *perpendicular to the horizon* are all equal to *each other* and to the time of falling down the *diameter*.

4. Suppose now that a body falls down a system of planes AB, BC, CD, &c. *inclined to each other*, and that no velocity is *lost* in falling from one plane to the other. Draw AF parallel to the horizon, and produce DC, CB to meet it in F, E; through BC draw Bb, Cc parallel to AF, and let fall FG at right angles to the horizon. By Art. 1, the velocity down AB = velocity down EB = velocity down Fb; ∴ the velocity down $AB + BC$ = velocity down $EB + BC$ (or EC) = velocity down FC = velocity down Fc; and reasoning in the same manner, the velocity down $AB + BC + CD$ = velocity down FD = velocity down FG; "i. e. the *whole velocity* acquired "by a body falling down a system of planes of this "kind (supposing no velocity *lost* at the points "B, C, &c.) is equal to the velocity which a body "would acquire in falling freely through their *joint* "*height* FG."

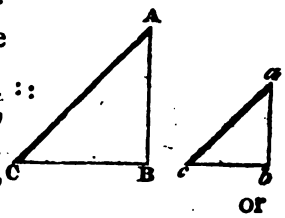


5. By Art. 1, page 189, Vol. I. *the velocity lost in passing from the plane* AB *to the plane* BC : *velocity acquired in falling down* AB :: *versed sine of* $\angle ABE$: *radius;* if therefore the angle $ABE = 0$ (and

(and consequently its *versed sine*=0); then the *velocity lost in passing from one plane to the other* will also be equal to nothing. Suppose now the number of the planes AB, BC, CD, &c. to be *increased* and their lengths and inclinations to each other to be *diminished* ad infinitum, then any two planes AB, BC will become *two contiguous points in a curve*, and consequently may be considered as *coinciding with a tangent to that part of the curve*; the angles ABE, ECF, &c. will therefore become equal to 0, and the *velocities lost* at the points B, C, &c. will also be equal to 0[a]; hence, "if a body "falls down a perfectly smooth "*curvilinear* plane AD, the "velocity acquired will be equal "to the velocity acquired in falling down its *perpen-* "*dicular height* AG."



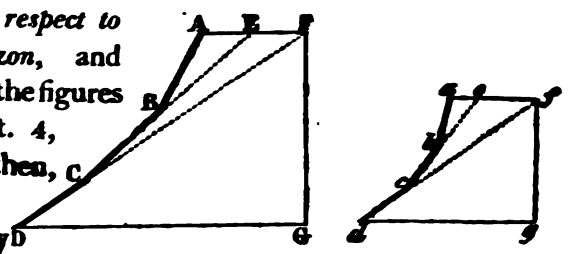
6. Let AC, ac be two planes *similarly* inclined to the horizon, then $AC : ac :: AB : ab$, and $\sqrt{AC} : \sqrt{ac} :: \sqrt{AB} : \sqrt{ab}$. Now the *time* down AC : the *time* down $ac :: \frac{AC}{\sqrt{AB}} : \frac{ac}{\sqrt{ab}} ::$ $\frac{AC}{\sqrt{AC}} : \frac{ac}{\sqrt{ac}} :: \sqrt{AC} : \sqrt{ac}$, or



(a) To this it may possibly be objected, that although the angles ABC, ECF, &c. (and consequently their *versed sines*) become

or $\sqrt{AB} : \sqrt{ab}$; and the *velocity* acquired in falling down AC : the velocity down $ac :: \sqrt{AB} : \sqrt{ab}$, or $\sqrt{AC} : \sqrt{ac}$; hence the *times* and *velocities* of bodies falling down planes *similarly inclined to the horizon* are to each other both as the square roots of the *lengths* and as the square roots of the *heights* of the planes.

7. Let there be two systems of planes AB, BC, CD, &c. ab, bc, cd, &c. *similar, and similarly situated with respect to the horizon*, and complete the figures as in Art. 4, page 16; then, supposing no velocity



to be lost in passing from one plane to the other, and using the notation of Exam. 12, page 11, we have, by Art. 6, T^{AB}

become equal to nothing, yet since the *radii* with which they are compared become also equal to nothing, the ratio of their *versed sines* to their *radii* (and therefore of the *velocity lost* to the *velocity down the planes*) will remain *finite*; but it should be recollected that the *versed sine* of an arc : its *chord* :: *chord* : *diameter*, and, in *very small* angles, the versed sine of an arc : *arc* :: *arc* : diameter; if therefore the *arc* vanishes with respect to the *diameter* or *radius*, the *versed sine* vanishes with respect to the *arc*; hence, in *very small* angles, the versed sine will vanish with respect to the radius, although the radius itself be indefinitely small; consequently the velocity *lost* will vanish with respect to the velocity *down the planes*.

$T^{AB} : T^{ab} :: \sqrt{AB} : \sqrt{ab}$ (X),

$T^{EC} : T^{ec} :: \sqrt{EC} : \sqrt{ec} :: \sqrt{AB} : \sqrt{ab}$,[a]

$T^{EB} : T^{eb} :: \sqrt{EB} : \sqrt{eb} :: \sqrt{AB} : \sqrt{ab}$;

Hence,

$T^{EC} - T^{EB}$ (or T^{BC}) : $T^{ec} - T^{eb}$ (or T^{bc}) :: $\sqrt{AB} : \sqrt{ab}$ (Y).

In the same manner it may be shewn, that

$$T^{CD} :: T^{cd} :: \sqrt{AB} : \sqrt{ab} \text{ (Z).}$$

From the proportions marked (X), (Y), (Z), therefore we have

$T^{AB+BC+CD} : T^{ab+bc+cd} :: \sqrt{AB} : \sqrt{ab}$ [b]

$:: \sqrt{AB+BC+CD} : \sqrt{ab+bc+cd}$

"i.e. the times of descent down similar systems of "inclined planes are as the *square roots of the* "*lengths of the planes.*[c]"

8. If the number of the planes AB, BC, CD, &c. ab, bc, cd, &c. be *increased* and their lengths and inclinations to each other be *diminished* ad infinitum, then the polygons $ABCD$, $abcd$, become *similar curves*,

(a) For by sim. Δ^s, $EB : eb :: AB : ab$; and from similar planes, $BC : bc :: AB : ab$; $\therefore EB + BC$ (or EC) : $eb + bc$ (ec) :: $AB : ab$; and $\sqrt{EC} : \sqrt{ec} :: \sqrt{AB} : \sqrt{ab}$. In the same manner it may be shewn that $FD : fd :: AB : ab$, or $\sqrt{FD} : \sqrt{fd} :: \sqrt{AB} : \sqrt{ab}$.

(b) For since $AB : ab :: BC : bc :: CD : cd$, $AB + BC + CD : ab + bc + cd :: AB : ab$; or $\sqrt{AB} : \sqrt{ab} :: \sqrt{AB+BC+CD} : \sqrt{ab+bc+cd}$.

(c) The *velocity* down $AB + BC + CD$: *velocity* down $ab + bc + cd$:: $\sqrt{FG} : \sqrt{fg} :: \sqrt{FD} : \sqrt{fd} :: \sqrt{AB} : \sqrt{ab} :: \sqrt{AB+BC+CD} : \sqrt{ab+bc+cd}$; the *velocities* as well as the *times* are therefore as the *square roots of the lengths of the planes.*

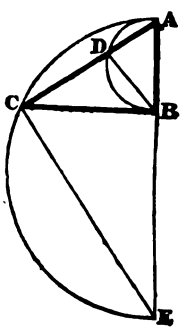
curves, in falling down which no velocity is *lost*; hence the *times* of descending through similar curves similarly situated with respect to the horizon are as the *square roots of the lengths of those curves*[a].

LI.

A few Questions relating to the Descent of Bodies upon Inclined Planes, demonstrated GEOMETRICALLY.

We shall conclude this Lecture with the solution of a few problems relating to the times of descent of bodies upon inclined planes *bearing certain given relations to each other*; they are matters of curiosity rather than of practical utility, but may serve to shew the manner in which the principles of *Geometry* can be applied to this branch of our subject.

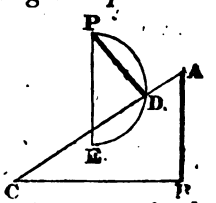
1. Let AC be an inclined plane whose length is AC and perpendicular height AB; draw CE at right angles to AC, and produce AB to meet it in E; upon AB, AE describe the semicircles ADB, ACE, and join DB, CE. By Art. 3, page 15, the times of falling down the *chords* AD, AC are equal to the times of falling down the *diameters* AB, AE respectively; hence the intersection (D) of the semicircle ADB with the



[a] Suppose the curves to be *circular arcs*; then, since similar circular arcs are to each other as the *radii* of the circles to which they belong, the *times* of descending through these arcs will be to each other as *the square roots of their radii*.

plane AC, determines the *part* (AD) which a body describes *whilst another body would fall freely through the height* AB[b]; and the intersection (E) of the perpendicular CE with AB produced, determines the *space* (AE) through which a body would fall freely *whilst another body describes the whole length of the plane* AC[c].

2. If it were required to determine the point D, "so that the time of falling from a *given point* P "to the plane shall be equal to the "time of falling through the *height* "AB;" then through P draw PE *parallel and equal to* AB, upon PE describe a semicircle cutting AC in D, and join PD; PD will be the line required. For the time down the *chord* PD = time down the *diameter* PE = time down AB (for PE is equal to AB).



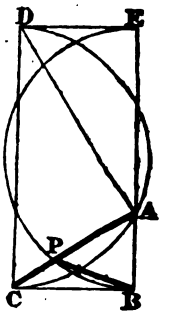
3. To find the point P, such that "the time "of falling from it to the *given* point B shall "be

(b) Since the *time down* DB = the time down AB, the intersection of the semicircle ADB with the plane AC determines also *that point of the plane* from which a body must fall to *the given point* B in the same time as a body would fall freely *through the height of the plane.*

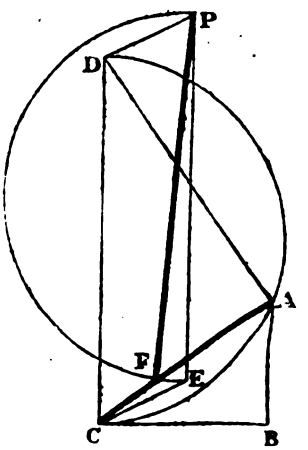
(c) Since ADB, ACE are *right angles*, (by Euc. 6. 8.) $AD = \frac{AB^2}{AC}$, and $AE = \frac{AC^2}{AB}$; $\therefore AD : AE :: \frac{AB^2}{AC} : \frac{AC^2}{AB} :: AB^3 : AC^3$:: cube of the *height* : cube of the *length.*

AB : AE

" be equal to the time of falling down AC the " *length* of the plane," draw CD at right angles to BC, and AD at right angles to CA; let them meet in D, and upon CD describe a semicircle which will pass through A; through D draw DE parallel to CB, and produce BA to meet it in E; upon BE describe a semicircle cutting AC in P; P is the point required. For the time down PB = the time down EB = the time down DC[a] = the time down AC.



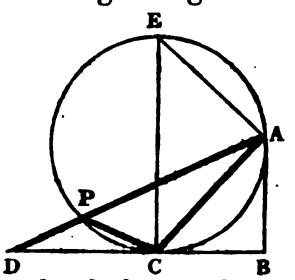
4. To determine the point F upon the plane AC, so " that the time of falling " from a *given point* P to it " shall be equal to the time " through AC," describe the semicircle DAC as in the preceding article, and join DP; complete the parallelogram $DCPE$, and upon PE describe a semicircle cutting AC in F; F is the point required. For the time down PF = the time down PE = the time down DC (since PE is equal to DC) = the time down AC.



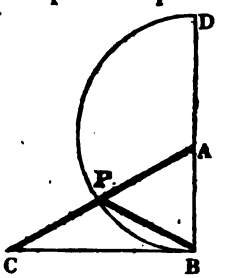
5. AC,

(a) $DCBE$ is a *parallelogram*; $\therefore DC = EB$, and consequently the *time* down EB = the *time* down DC.

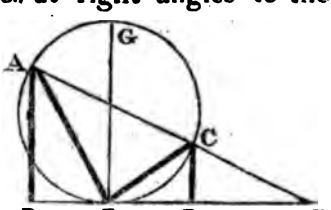
5. AC, AD are two inclined planes having the *common altitude* AB; draw CE at right angles to CB, and AE at right angles to CA; let them meet in E, and describe a circle (whose *diameter* is CE) cutting AD in P; then P is the point of the *plane* AD from which a body will fall to the given point C *in the same time that another body falls down the plane* AC. For join PC, then the time down PC = the time down EC = the time down AC.



6. Let it be required to determine upon the plane AC a point P, such that the time of falling from P to B shall be equal to some *given time* T. Draw BD at *right angles* to BC and *equal* to mT^2 (= space described by gravity in the time T); upon BD describe a semicircle cutting AC in P; P is the point required. For the time down PB = time down $DB = T$.

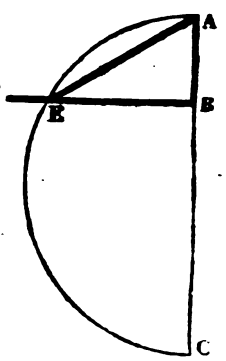


7. AB, CD are two lines at right angles to the horizon placed at a given distance (BD) from each other; join AC, and let BD, AC, when produced, meet in the point E; then if F be taken, such a point that $EF^2 = AE \times EC$, and AF, CF be



be joined, *the time down* AF shall be equal to *the time down* CF. For since $EF^2 = AE \times EC$, a circle may be described which shall pass through the points A, C, and *touch* BD in F (Euc. 3, 37); draw FG at right angles to BD, and it will be the *diameter;* and the time down AF = the time down GF = the time down CF.

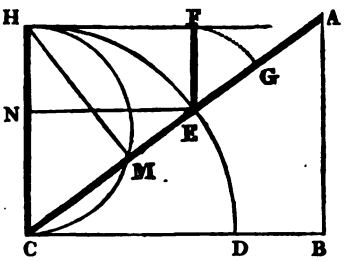
8. A body falls down AB, and then describes BD ($=2AB$) in the same time with the velocity acquired continued uniformly; it is required to find a point E in the line BD, such that *the time down the plane* AE shall be equal to *the whole time of the body's motion from* A through B to D. Produce AB to C, making $BC = 3AB$; upon AC describe a semicircle cutting BD in E; E is the point required. For join AE, then the time down AE = the time down AC = the time down $4AB$ = *twice* the time down AB = the time of descending through AB + the time of describing $2AB$ (or BD) with the velocity in B continued uniformly [a].



9. HD is a *quadrantal arc*, whose center is C, and CH perpendicular to the horizon; AC is a *plane given*

(a) $AE^2 = AB \times AC = AB \times 4AB = 4AB^2$, $\therefore AE = 2AB$, or $AE : AB$ (:: rad. : sin. $\angle AEB$) :: 2 : 1; hence $\angle AEB = 30^\circ$.

given in position; through H draw HF parallel to CD, and through E draw EF parallel to CH; with radius EF describe the circular arc FG; then if a body falls through HC, and is projected along the plane CA with the velocity acquired, it will describe upon the plane *a space* (CG) *equal to* $HC+EF$, in the same time that it has been falling freely through HC. Upon HC describe a semicircle cutting AC in M; join HM, and complete the parallelogram $HFEN$; then $EM = NH$[b] $= EF = EG$; $\therefore CG = (CE + EG =)$ $CE + EM$. Now MC is the space through which a body would fall *down the plane* whilst another body *descends freely through* HC; and $2HC$ or $2CE$ is the space which a body would describe upon the plane in the same time with the velocity acquired (in falling through HC) *continued uniformly*; a body therefore projected up the plane with the velocity acquired through HC would have a *progressive* tendency, which is measured by $2CE$, and a *retrograde* one, which is measured by MC; hence the *whole space* which it would describe upon the plane would be $2CE - MC$ or $CE + EM$[c] $= CG = HC + EF$.

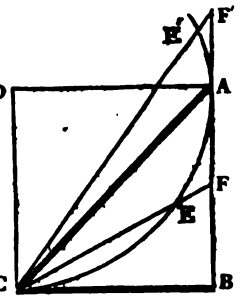


10. Sup-

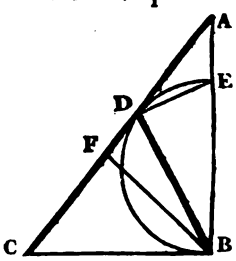
(b) $EM = NH$, for they are *versed sines of the same arc* HE.

(c) $2CE - MC = CE + \overline{CM + ME} - MC = CE + EM$.

10. Suppose the *base* BC of an inclined plane to be *given*, and it was required to determine the *height*, so that the time down the plane should be the *least* possible. Draw BF at right angles to CB, and take $BA=BC$; complete the square $ABCD$, and with center D describe the circle CBE', which will evidently *touch* BF', BC in the points A, C; join AC, and draw any other chords CE, CE', which produce to meet BF' in F, F'. The time down $E'C$ = time down AC = time down EC, ∴ the time down $F'C$ or FC (being greater than the time down $E'C$ or EC) is *greater* than the time down AC; hence AC is the plane of *quickest descent* from BF to C[a].



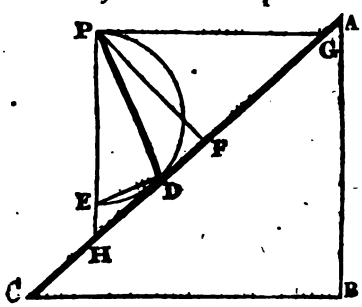
11. If it be required to find the line of quickest descent from the plane AC to B, take $CD=CB$, and join DB; then DB will be the line required. For draw DE at right angles to DB, and upon EB describe the semicircle EDB, which will *touch* BC,



(a) Since $AB=BC$, it appears that when the *base* of a plane is given, its *height must be equal to the base* for the time down the plane to be a *minimum*. This may also be shewn *fluxionally*, thus; Let $BC=a$, $AB=x$, then $AC=\sqrt{a^2+x^2}$; but $T\propto\frac{L}{\sqrt{H}}$, $\therefore T^2\propto\frac{L^2}{H}\propto\frac{AC^2}{AB}$; hence $\frac{a^2+x^2}{x}$ or $\frac{a^2}{x}+x=min.$, and $-\frac{a^2\dot{x}}{x^2}+\dot{x}=0$; from which $x=a$.

BC, CA in B, D[b]; then since no other line but BD can be drawn from B to the plane AC, which does not *cut* the circle EDB, DB will be the line of *quickest descent* from AC to B.

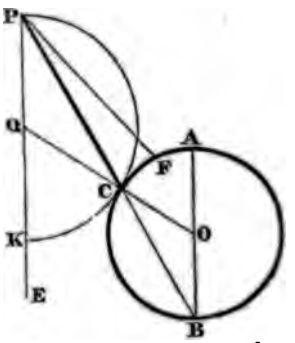
12. To find the line of quickest descent from a *given point* P *to the plane* AC, draw PG parallel to CB, and PH parallel to AB; take $GD = GP$, and join PD; PD is the line required. For draw DE at right angles to PD, and upon PD describe the semicircle PDE, which (for the same reason as in the preceding article) will *touch* AC, PG in D, P; and as no other line but PD can be drawn from P to the plane AC, which does not *cut* the circle, PD will be the line of *quickest descent* from P to AC.



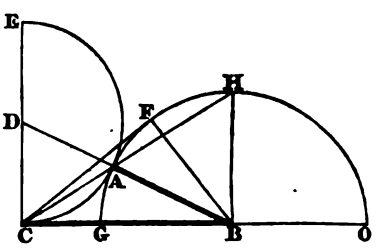
13. Let it next be required to find the line of quickest descent *from a given point* P *to the circle* ACB. Take O the *center* of the circle ACB, draw the diameter AB perpendicular to the horizon, and

(b) CB touches the circle in B, because BA is at *right angles* to BC, and CD is equal to CB; but if two equal lines be drawn from a given point to *meet* a given circle, and *one* of them *touch* the circle, it may easily be proved that the *other touches it also*.

and PE parallel to AB; join PB, and let it cut the circle in C; join OC, and produce it till it cuts PE in Q; then the triangles PQC, COB will be similar, and since $OC = OB$, QC will be equal to QP; a circle (PCK) therefore described with *center* Q and *radius* QC will pass through P, and since OQ *joins the centers* of the two circles ACB, PCK, it will pass *through the point of contact*; hence, as the circle PCK touches the circle ACB in the point C, it can touch it in no other point; and since no other line but PC can be drawn from P to the circle ACB, which does not *cut* the circle PCK, PC will be the line of *quickest descent* from P to the circle ACB.



14. Lastly, suppose two sides of a triangle (one of which is *parallel to the horizon*) were given, and it was required to determine the *third side*, so that the time of falling down it should be a *minimum*. Let BC represent the *longer* of the two given sides, and take BG equal to the *shorter* of them; with center B and radius BG describe the semicircle GHO; draw BH at right angles to CB, and join CH;



CH; the point A where CH cuts the semicircle GHO determines the position of the *shorter side* AB, so that the time down the third side AC shall be a *minimum*. For draw CE at right angles to CB, and produce BA to meet it in D; then (as in the preceding article) it may be shewn that if with center D and radius DC a circle (CAE) be described, it will touch the semicircle GHO in the point A; and and as no other line but CA can be drawn from C to the semicircle GHO, which does not *cut* the circle CAE, the time down AC will be the *least* when the *shorter* of the two given sides is in the position AB. In this case the *length* of the side AC is also very easily determined; for by the property of the circle $CA \times CH = CO \times CG$; now $CH = \sqrt{CB^2 + BH^2} = \sqrt{CB^2 + BA^2}$; $CO = CB + BO = CB + BA$; and $CG = CB - BG = CB - BA$; hence $CA \times \sqrt{CB^2 + BA^2} = \overline{CB + BA} \times \overline{CB - BA} = CB^2 - BA^2$, $\therefore CA = \dfrac{CB^2 - BA^2}{\sqrt{CB^2 + BA^2}}$.

Lecture

Lecture XIII.

ON THE ASCENT AND DESCENT OF BODIES CONNECTED TOGETHER BY A CORD GOING OVER A FIXED PULLEY.

THE subject of this Lecture may be divided into two parts. I. When the bodies thus connected ascend or descend in a direction *perpendicular to the horizon*. II. When one or both of the bodies ascend or descend upon *inclined planes*.

LII.

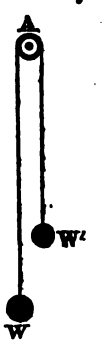
On the perpendicular *ascent or descent of bodies over a fixed Pulley.*

1. It has already been observed, in the First Lecture, that the *momentum* of a body is proportional to *its velocity and quantity of matter*; the *velocity* therefore varies as the *momentum* directly and the *quantity of matter* inversely. Now the *force* which generates *momentum* in a body is called the *moving force*, and the force which generates *velocity* is called the *accelerative force* (*); if therefore,

(*) The *accelerative* force of a body will therefore be measured by the *velocity* generated in a given time.

fore, for the momentum and velocity of a body in motion, we substitute the *causes which produce them*, it may be said, that the *accelerative force* is as the *moving force* directly, and the *quantity of matter moved* inversely; or in general, if $M =$ *moving force* of a body, $Q =$ its *quantity of matter*, and $F =$ the *accelerative force*, then $M \propto F \times Q$, and $F \propto \frac{M}{Q}$.

2. Let us apply these expressions to the case of *two weights* W, W' (of which W is the *greater*), acting upon each other by means of a perfectly flexible cord (WAW') passing over a small *fixed pulley* (A); supposing the *weight* of the cord to *be inconsiderable* with respect to W, W', and that the pulley moves quite freely round its axis. It is evident, that the motion will take place in the direction of the *greater* body (W), and that the *moving force* is proportional to the *difference of the weights* ($W - W'$); the *quantity of matter* moved is $W + W'$; hence the *accelerative* force of W (which varies as $\frac{M}{Q}$ by Art. 1.) will be proportional to $\frac{W - W'}{W + W'}$; i. e. the accelerative force of W, when it moves *in connection with* W': its accelerative force when it moves *freely* :: $\frac{W - W'}{W + W'}$



$: \frac{W^{(a)}}{W} :: \frac{W-W'}{W+W'} : 1$; supposing therefore that the force of *gravity* is represented by *unity*, the force which accelerates W will be measured by the *fraction* $\frac{W-W'}{W+W'}$; and since W, W' are *constant* quantities, this force will be a *constant* force, producing similar effects to those which the *force of gravity* produces[(b)].

3. Having

(a) When W moves *freely*, the *moving force* is *its own weight*; the *quantity of matter* moved is *also* its own weight; $\therefore$ the *accelerative force* will be measured by $\frac{W}{W}$ or *unity*.

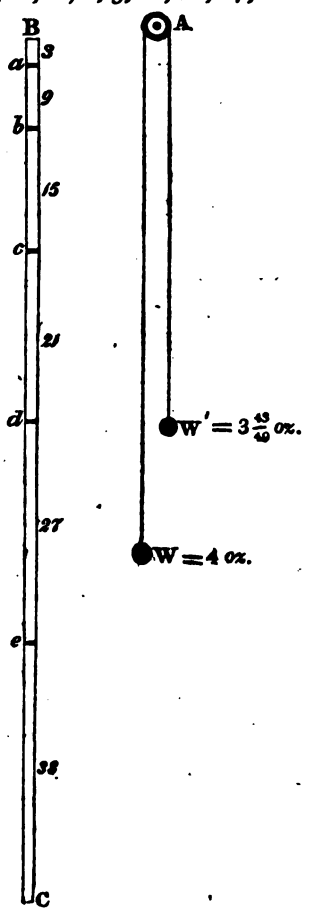
(b) Indeed, by means of two weights connected together in this manner by a cord going over a fixed pulley, we might make the *law of acceleration* of the force of gravity visible to an observer. For instance, let $W=4$ *oz.* $W'=3\frac{43}{49}$ *oz.*; then $\frac{W-W'}{W+W'}=\frac{4-3\frac{43}{49}}{4+3\frac{43}{49}}=\frac{196-190}{196+190}=\frac{3}{193}$; the force which accelerates W is therefore $\frac{3}{193}$d part of the *force of gravity*; and as a body in the first second of its fall describes $16\frac{1}{12}$ feet, or 193 inches, by the force of gravity, W in the first second of its descent will describe a space equal to $(\frac{3}{193}\times 193$, or$)$ 3 *inches*[†]; for the same reason, in *six* seconds it

* We assume $W'=3\frac{43}{49}$ *oz.* for the purpose of getting $\frac{W-W'}{W+W'}$, such a fraction as shall give the *space* described by W in 1″ *a definite number of inches*.

† For by Art. 2, page 4, $S=mFT^n=$ (since $m=193$ inches, $F=\frac{3}{193}$, and $T=1$) $193\times\frac{3}{193}\times 1=3$ inches.

3. Having thus calculated the *accelerative force* with

$\frac{3}{193} \times 193 \times 36$ or 108 inches = 9 feet; and in the *successive* seconds, beginning from the first, it would describe (spaces which are as the odd numbers 1, 3, 5, 7, 9, 11, or) 3, 9, 15, 21, 27, 33 inches. Suppose now a white rod *BC* to be marked in the points *a*, *b*, *c*, *d*, *e*, in such manner that *Ba*, *ab*, *bc*, *cd*, *de*, *eC*, are respectively 3, 9, 15, 21, 27, 33 inches; and let it be so placed with respect to the weights going over the pulleys, that the eye of the observer may command both the weight and the rod whilst *W* descends from rest through 9 feet, beginning at the fixed pulley *A*. At the end of the *first* second he would observe it pass the point *a*; at the end of the *second*, the point *b*; and so on, until it arrived at the point *C* at the end of the *sixth* second; thus describing *spaces* which are as the *squares of the times*, and descending in a manner altogether analogous to that of a body falling *freely* by the *force of gravity*.



We have said thus much just for the purpose of explaining the principle upon which the law of acceleration of the force of gravity may be made evident to an observer; but for a full description of the apparatus necessary for exhibiting all the phænomena attending it, and for a variety of luminous experiments upon the subject, the reader is referred to the *7th Section of Mr. Atwood's Treatise on the Rectilinear and Rotatory Motion of Bodies.*

with which the weight W descends; the *space* described by it in a given time, the *velocity* acquired at any point of its descent, and the *time* of descending from any given altitude, may be ascertained from the principles laid down in Art. 2. page 3; of which the following are examples.

EXAM. I.

A weight of 4 lbs is attached to a weight of one lb. by means of a cord going over a fixed pulley; How far will the heavier weight descend in 3″?

Here $W=4$, $W'=1$: $\therefore F=\frac{W-W'}{W+W'}=\frac{4-1}{4+1}=\frac{3}{5}$; hence, by Art. 2, page 3.

$S\,(=mFT^2)=\frac{3}{5}\times 16\frac{1}{12}\times 9=86.76$ ft.

EXAM. II.

Two weights of one pound each are suspended by a cord going over a fixed pulley; an ounce weight is added to one of them; How long will it be in descending through 12 feet? and what velocity will it have acquired at the end of its descent?

Here $W=17$ oz., $W'=16$: $\therefore F=\frac{1}{33}$; now $T\left(=\sqrt{\frac{S}{mF}}\right)$

$=\sqrt{\frac{12}{16\frac{1}{12}\times\frac{1}{33}}}=4.96$ *seconds*;

$V=2mFT=32\frac{1}{6}\times\frac{1}{33}\times 4.96=4.83$ feet in 1″.

EXAM.

EXAM. III.

From the top of a tower 50 *feet high, a stone weighing* 2 *lbs is let fall from rest; at the same instant a weight of* 6 *lbs, having another weight of* 5 *lbs attached to it by a cord going over a fixed pulley, is let fall from the same point; Compare the times of descent of the stone and the weight, and the momenta with which they respectively strike the earth.*

By Art. 2, page 3, $S = mFT^2$, $\therefore T^2 = \frac{S}{mF} \propto \frac{S}{F}$ and if S *be given*, then $T^2 \propto \frac{1}{F}$, and $T \propto \frac{1}{\sqrt{F}}$; now if the *force of gravity* $= 1$, then the *accelerative force* of the *weight* $= \frac{6 - 5\frac{1}{2}}{6 + 5\frac{1}{2}} = \frac{1}{23}$; hence the time of descent of the *weight* : time of descent of the *stone* :: force of *gravity* : *accelerative force* of the weight $:: 1 : \frac{1}{\sqrt{23}} :: \sqrt{23} : 1$.

Again, $V = 2\sqrt{mFS}$, and when S *is given* $V \propto \sqrt{F}$; $\therefore$ velocity acquired by the *weight* : velocity acquired by the *stone* $:: \frac{1}{\sqrt{23}} : 1 :: 1 : \sqrt{23}$; hence (since the *momentum* $\propto Q \times V \propto W \times V$) the *momentum* of the weight : *momentum* of the stone $:: 6 \times 1 : 2 \times \sqrt{23} :: 3 : \sqrt{23}$.

EXAM. IV.

Two equal *weights are connected together as in the preceding examples; what weight must be added to one*

one of them, to make it acquire, in falling for 6", a velocity of 48 feet per second; and through what space will it have fallen in that time?

Let W = either of the equal weights, and x = the *weight required*; then the *accelerative force* of $W+x$ $\left(=\frac{W+x-W}{W+x+W}\right)=\frac{x}{2W+x}$.

Now $V=2mFT$, $\therefore F=\frac{V}{2mT}$; in the present instance, $V=48$, $T=6$, and $F=\frac{x}{2W+x}$; hence $\frac{x}{2W+x}=\frac{48}{12m}=\frac{4}{m}$, from which $mx=8W+4x$, or $x=\frac{8W}{m-4}=\frac{8W}{16\frac{1}{12}-4}=\frac{96}{145}W$.

Since therefore the weight to be *added* to W is $\frac{96}{145}W$, the force which *accelerates the greater weight* will be $\frac{145-96}{145+96}=\frac{49}{241}$; now $S\ (=mFT^2)=16\frac{1}{12}\times\frac{49}{241}\times 36=117.6$ feet for the space descended through in *six* seconds by a body falling under these circumstances.

EXAM. V.

If the weight W, instead of descending from rest, has a velocity of (a) feet in a second communicated to it at the beginning of its fall, it is required to ascertain the velocity (V) which it will have acquired after T'' are elapsed.

Since

Since a *velocity* of (a) feet in 1″ is communicated to W, its *momentum* (for $M \propto Q \times V$) will be increased by Wa; but this momentum or *moving force* is applied to the mass $W + W'$, the *velocity* communicated to this mass therefore $\left(\text{for } V \propto \frac{M}{Q}\right)$ will be $\frac{Wa}{W+W'}$. Now the velocity generated by the *accelerative force* in the time T $(= 2mFT)$ $= \frac{\overline{W-W'}.2mT}{W+W'}$; hence the velocity after T'', when the weight W is impelled downwards with a velocity of (a) feet in 1″, is $\frac{Wa+\overline{W-W'}.2mT}{W+W'}$.

Cor. The *space* described under the same circumstances is also readily ascertained; for the space described in T'' with the *uniform velocity* $\frac{Wa}{W+W'}$ is $\frac{WaT}{W+W'}$; and the space described in T'' with the velocity of *acceleration* $(= mFT^2) = \frac{\overline{W-W'}.mT^2}{W+W'}$;

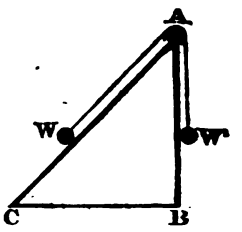
$\therefore$ the *whole space* described $= \frac{WaT+\overline{W-W'}.mT^2}{W+W'}$.

LIII.

On the motion of two bodies over a Pulley, when one, or both, of them descends or ascends upon an inclined plane.

1. If

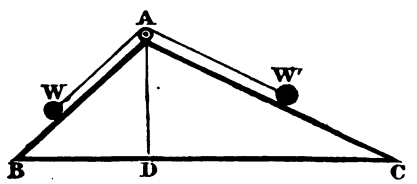
1. If two bodies (W, W') are connected together by a cord going over a fixt pulley (A), and one of them (W) descends upon an *inclined plane* (AC), then the *moving force* of W descending *freely* down the plane $= \frac{W \times H}{L}$ (H being the *height*, and



L the *length* of the plane); hence the *moving force of W when connected with W'* $= \frac{W \times H}{L} - W' = \frac{WH - W'L}{L}$; and the *accelerative force* $= \frac{WH - W'L}{L.W + W'}$

2. If W' *draws W up the plane*, the moving force with which W' descends $= W' - \frac{WH}{L} = \frac{W'L - WH}{L}$; and the *accelerative force* $= \frac{W'L - WH}{L.W + W'}$; if $W' = W$, then the accelerative force of $W' = \frac{L - H}{2L}$.

3. Let *both* bodies move upon inclined planes, having the common height AD (H), and let W be the *descending* body; let $AB = L$, $AC = L'$; then the moving force of W, descending *freely* down the plane



(*) For by Art. 1, page 2, the *accelerative force* of W down the plane $AC = \frac{H}{L}$; and by Art. 1 of this Section, $M \propto F \times Q$, $\therefore$ the *moving force* of W down the plane $= \frac{W \times H}{L}$.

plane $AB=\frac{WH}{L}$; and the moving force of W' descending *freely* down the plane $AC=\frac{W'H}{L'}$; hence the *moving force of W, when connected with W'* $=\frac{WH}{L}-\frac{W'H}{L'}=\frac{\overline{WL'-W'L}.H}{LL'}$; $\therefore$ the *accelerative* force of W down the plane $AB=\frac{\overline{WL'-W'L}.H}{LL'.\overline{W+W'}}$.

If $W'=W$, then this force $=\frac{\overline{L'-L}.H}{2LL'}$ [b]

EXAM. I.

A weight of 10*lbs* (W), *placed upon an inclined plane whose height is* $\frac{1}{3}d$ *of its length, is connected with another weight* (W') *of* 3 *lbs, hanging perpendicularly* (*as in Fig. Art.* 1); *How far will W descend in* 8″?

Here

(b) Let $\angle ABD=\alpha$, $\angle ACD=\beta$; then $L:H::$ rad. (1) : sin. α, and $L':H::$ rad. (1) : sin. β, $\therefore\frac{H}{L}=$sin. α, and $\frac{H}{L'}$ =sin. β; hence the *moving force* of W down the plane $AB=W\times$ sin. α, and the *moving force* of W' down the plane $AC=W'\times$ sin. β; consequently the *accelerative force* of W (when connected with W') down the plane $AB=\frac{W\times\text{sin. }\alpha-W'\times\text{sin. }\beta}{W+W'}$; and if $W'=W$, then this force $=\frac{\text{sin. }\alpha-\text{sin. }\beta}{2}$; but the expressions in the text are by far the most convenient for estimating the motion of bodies thus connected together upon inclined planes, *if their heights and lengths be given in whole numbers.*

Here $W = 10$, $W' = 3$, $H = 1$, $L = 3$, $T = 8$; $\therefore$ by Art. 1, the accelerative force of W $\left(= \frac{WH - W'L}{L.W + W'}\right) = \frac{10 \times 1 - 3 \times 3}{3.10 + 3} = \frac{1}{39}$; hence $S\ (= mFT^2) = 16\frac{1}{12} \times \frac{1}{39} \times 64$ $= 26.38$ feet.

EXAM. II.

In what time will a weight hanging perpendicularly, draw an equal *weight up an inclined plane whose length is* 12 *feet, and height* 10 *feet?* (*See Fig. in Art.* 1.)

This is a case of Art. 2, where $W' = W$; $\therefore$ the accelerative force of the weight hanging *perpendicularly* $\left(= \frac{L - H}{2L}\right) = \frac{12 - 10}{24} = \frac{1}{12}$; hence $T = \sqrt{\frac{s}{mF}} = \sqrt{\frac{12}{16\frac{1}{12} \times \frac{1}{12}}} = 2.9$ seconds = the time in which W' would draw W up the whole length of the plane.

EXAM. III.

A weight of 5 *lbs draws a weight of* 6 *lbs up an inclined plane whose elevation is* 30°; *what velocity will the descending body have acquired, after it has been falling for* 6″? (*Fig. in Art.* 1.)

Here $W = 6$ lbs., $W' = 5$, $H = \sin. 30^\circ = \frac{1}{2}L$; $\therefore$ by Art. 2, $F = \left(\frac{W'L - WH}{L.W + W'} = \right)$ $\frac{5L - 3L}{11L} = \frac{2}{11}$; hence $V = (2mFT =)$ $32\frac{1}{6} \times \frac{2}{11} \times 6 = 35\frac{1}{11}$ feet in 1″ = velocity of W' after it has descended for 6″.

EXAM.

EXAM. IV.

Given the height of an inclined plane, to find its length, so that a given weight descending vertically shall draw another given weight up it in the least time possible.

Let H = the *height*, x = *length* of the plane; then, by Art. 2, the accelerative force of W' (Fig. in Art. 1.) $= \frac{W'x - WH}{x.\overline{W+W'}}$, which *varies* as $\frac{W'x-WH}{x}$. Now $T^2 = \frac{S}{mF} \propto \frac{S}{F}$, $\therefore$ T^2 *through the space* x [b] $\propto \frac{x}{F} \propto \frac{x^2}{W'x - WH}$, which is to be a *minimum*; hence $\frac{W'x-WH}{x^2}$ or $\frac{W'}{x} - \frac{WH}{x^2}$ = *maximum*; from which we have $-\frac{W'\dot{x}}{x^2} + \frac{2WH\dot{x}}{x^3} = 0$, or $x = \frac{2WH}{W'}$ = *length of the plane.*

Cor. Hence the *length* of such a plane : its *height* :: $2W$: W'. If $W' = W$, then the *length* : *height* :: 2 : 1; from which it follows that the *elevation* of a plane, up which a given weight would draw an *equal* weight in the *least time possible*, is 30°.

EXAM. V.

In the Fig. of Art. 3, *page* 38, *the common height* (AD) *of the two planes is* 20 *feet*; AB = 30 *feet*; AC = 40 *feet*; *supposing the weights to be equal, how long will* W *be in descending down the plane* AB.

This

(b) It is evident that the weight W' must descend through a space equal to AC (or x) whilst it draws the weight W up the whole length of the plane.

This is a case of Art. 3, where $W' = W$; $\therefore$ the accelerative force of W down the plane AB $\left(= \frac{L' - L.H}{2LL'}\right) = \frac{40 - 30.20}{2 \times 30 \times 40} = \frac{1}{12}$; hence $T = \sqrt{\frac{S}{mF}} = \sqrt{\frac{30}{16\frac{1}{12} \times \frac{1}{12}}} = 4.7$ seconds = *the whole time of descent of W down the plane AB.*

EXAM. VI.

In the Figure just now adverted to, suppose the weights W, W′ to be equal to each other, and that the inclination of the plane AB is 60°, *and of the plane AC* 50°; *what proportion does the force which accelerates W down the plane AB bear to the force of gravity?*

By Note (*b*), p. 39, this force $= \left(\frac{\sin. \alpha - \sin. \beta}{2} =\right)$ $\frac{\sin. 60° - \sin. 50°}{2} = \frac{.8660 - .7660}{2} = \frac{.1000}{2} = .05$ $= \frac{1}{20}$*th part of the force of gravity.*

EXAM. VII.

In the Fig. of Art. 3, *page* 38, *given the height and length of the plane AB, to find the length of the plane AC, so that a given weight (W) may draw another given weight (W′) up the plane AC, in the least time possible.*

Let $AD = H$, $AB = L$, $AC = x$; then, by Art. 3, the accelerative force of W down the plane AB =

$= \left(\frac{WL' - W'L.H}{LL'.W + W'} =\right) \frac{Wx - W'L.H}{Lx.W + W'}$, which *varies* as $\frac{Wx - W'L}{x}$. Now $T^2 \propto \frac{S}{F}$, $\therefore$ T^2 *of describing the space* x *in direction of the plane* AB (for it is evident that W must descend through a space equal to AC whilst it draws W' up that plane) $\propto \frac{x^2}{Wx - W'L}$, which is to be *a minimum;* $\therefore$ $\frac{Wx - WL}{x^2}$ or $\frac{W}{x} - \frac{W'L}{x^2}$ = a *maximum*, or $-\frac{W\dot{x}}{x^2} + \frac{2W'L\dot{x}}{x^3} = 0$; from which $x = \frac{2W'L}{W}$.

Cor. Hence AC (x) : AB (L) :: $2W'$: W[a]. If $W' = W$, then AC : AB :: 2 : 1; $\therefore$ for the weight W to draw an *equal* weight up the plane AC in the *least time possible*, the length of AC must be *double* the length of AB.

LIV.

QUESTIONS FOR PRACTICE.

1. A weight (W) of 7 lbs is attached to another weight (W') of $5\frac{1}{2}$ lbs, by means of a cord going over a fixed pulley; How far will W descend from rest in 5″; and what velocity will it have acquired at the end of its fall?

Answer, $S = 48\frac{1}{4}$ feet; $V = 19\frac{3}{10}$ feet in 1″.

2. An

(a) Since sin. B : sin. C :: AC : AB (:: $2W'$: W), sin. $C = \frac{W \times \text{sin. } B}{2W'}$. If $W' = W$, then sin. $C = \frac{1}{2}$ sin. B. If $B = 90^\circ$, then sin. $C = \frac{1}{2}$ sin. $90^\circ = \frac{1}{2}$ radius = sin. 30°, which coincides with what was said in the *corollary* to Exam. 4.

2. An empty bucket (weighing 6 lbs), and a full one (weighing 25 lbs), are connected together by a cord going over a pulley fixed at the top of a well 166 feet deep; the length of the cord is so adjusted, that when the *full* bucket begins to descend from the top, the *empty* one begins to ascend from the bottom; How long will the full bucket be in drawing up the empty one to the top of the well?

Answ. 4.1 seconds.

3. Two *equal* weights (W) are suspended over a fixed pulley; what weight must be added to one of them, that it may be made to descend through 50 feet in 6″?

Answ. $\frac{100}{529}W$.

4. A weight of 5 lbs draws up one of 4 lbs, over a fixed pulley; at the instant of letting go the weight of 5 lbs, a *velocity of* 3 *feet in* 1″ is communicated to it; How far will it descend in 5″; and what velocity will it have acquired at the end of that time? *See Exam.* 5, *page* 36, *and Corollary.*

Answ. $S = 53\frac{1}{16}$ feet; $V = 19\frac{29}{31}$ feet in 1″.

5. A weight (W') of 10 lbs, hanging *perpendicularly*, draws a weight (W) of 15 lbs up an *inclined plane* whose elevation is 30°; How far will W' descend in 5″; and what velocity will it have acquired at the end of its fall?

Answ. $S = 40\frac{1}{24}$ feet; $V = 16\frac{1}{12}$ feet, in 1″.

6. There

6. There are two planes having a *common height* (as in Fig. Art. 3, page 38); the *length* of one plane is *three times*, and of the other *twice* the height; How long will a weight of 4lbs (descending down the *shorter* plane) be in drawing a weight of 5lbs *twenty feet* up the *longer plane*?

Answ. 5.7 seconds.

7. In the Fig. of Art. 3, page 38, the elevation of the plane AB is 30°; What must be the elevation of the plane AC, that a weight of 7lbs, moving down AB, may draw a weight of 8lbs up the plane AC in *the least time possible*? *See Note* (*a*), *page* 43.

Answ. 12°. 38′.

8. A weight of 2lbs (W), descending upon a plane whose elevation is 45°, draws another weight (W') of *one lb.* up a plane whose elevation is 30°; What proportion does the force *which accelerates* W bear to the force of *gravity*? Answ. $2\sqrt{2}-1 : 6$.

Lecture XIV.

ON THE ROTATORY MOTION OF BODIES ABOUT A FIXED AXIS.

FROM the manner in which the two bodies w supposed to act upon each other in the prece Lecture, it appears that their motions were formed in the direction of a straight line pas through their centers of gravity; in which case evident, that each particle of the whole mass, thus in motion, was equally accelerated[a]. The expres there adopted for the *accelerative force* of the scending body $\left(\text{viz. } \frac{\textit{the moving force}}{\textit{quantity of matter moved}}\right)$ only true when each particle of the mass moves w

(a) By attentively considering the manner in which two bo connected together by a cord going over a fixed pulley, per their motions; it is evident, not only that the *two bodies*, but *each particle* of the two bodies, move with the same velocity, *the velocity with which their respective centers of gravity mo*

the *same velocity*[b]; if therefore one of the bodies be impelled by some force which shall produce a *rotatory motion* about a fixed axis (in which case each particle of that body will move with a velocity *proportional to its distance from the axis*[c]), this expression for the accelerative force cannot be applied, till it has undergone an *alteration*, the nature of which we now proceed to explain.

LV.

On the method of estimating the Accelerative Force with which a body descends perpendicularly, when resisted by the inertia arising from the Rotatory Motion of a body, or system of bodies, about a fixed axis.

1. In the annexed Figure, *A* represents a *pulley* or *small wheel* whose plane is perpendicular to the *horizontal axis BC*; p, p', p'', p''', are small equal particles

(b) This expression is derived from the general principle, that "*the quantity of motion in a body varies as the quantity of matter into its velocity*," which is evidently founded upon the supposition that each particle of that quantity of matter is endued with the *same velocity*.

(c) For each particle will describe the circumference of a circle whose radius is *the distance of that particle from the axis*; and as the *angular* velocity of all the particles is the *same*, their *linear* velocity will be as the *circumferences* (i. e. as the radii) of the circles which they respectively describe.

particles of matter lying in a plane *parallel to* the *plane of the wheel*, and connected with each o and with the axis by the inflexible rods pp'', *void of gravity*, the distances Sp, Sp', Sp'', Sp''' from the axis being all equal to each other and to *the radius of the wheel A*. Suppose now that a *rotatory* motion be communicated to the particles p, p', p'', p''' by the descent of the *weig* attached to a cord going round the wheel A[a]; since the weight of the particles p, p'' placed at extremities of the rod pp'', and of the particles p' placed at the extremities of the rod $p'p'''$, *coun act*[b] each other, the *weights* of the particles



(a) It is proper to state here, that, *throughout this Lecture weight* of the wheel A and of the cord AP, and the *fr* of the axis upon the points of support at B, C, are suppos be so small as to render it unnecessary for them to be taken consideration.

(b) It is evident that the weight of any particle (p), *descends* in the course of its rotation, will tend to *increas* moving force of P, and that the weight of the *op* particle (p'') as it *ascends* will tend to *diminish* it in the *degree;* the same may be said of the particles p', p'''; it we shall, in every instance, suppose the *rotatory matter* so disposed, that *equal quantities* of it may lie at *equal opposite* distances from the axis of motion. If this be n case, the moving force of P will not be *constant*, but subj continual variations, from the *irregular action* of the different of the revolving system.

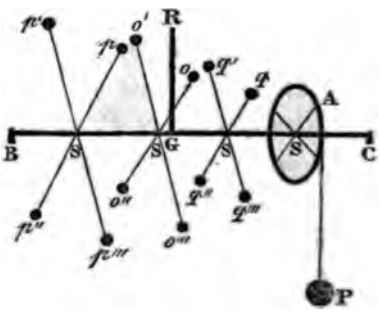
not at all affect the *moving force* of P, and the only effect produced by them will be to increase the *inertia*[c] of the system. Moreover, since these particles are all placed at a distance from the axis equal to the *radius of the wheel*, they must all move with the *same velocity* as any point A in the circumference of the wheel, i. e. with *the velocity of* P[d]; here then is a case of *rotatory* motion combined with the descent of P, in which *each particle of the whole mass moved* is *equally accelerated*; the *accelerative force* of P therefore may be estimated, as in former instances, "*by dividing the moving force by the quantity of matter*;" now the *moving force* is P, and the *quantity of matter moved* $P+p+p'+p''+p'''$; let $p+p'+p''+p'''=u$, then *the force which accelerates* P *in its descent* $=\frac{P}{P+u}$, the force of gravity being represented by *unity*.

2. Let

(c) The *inertia* of a body or system of bodies, as distinguished from their *gravity*, has been defined to be "that property which "all bodies possess of resisting the action of any force applied "to move them, whatever be the direction in which that force "acts;" and it is found, by experience, to be proportional to the *quantity of matter* contained in the bodies. (See Note III. Vol. I.) In the present instance, the *gravity* of the particles p, p', p'', p''' is *counteracted* by their position with respect to each other in the revolving system; but their *inertia*, or "*disposition to resist the force of P as it descends over the wheel A*," still remains.

(d) Since every point of the cord AP comes successively in contact with the wheel as P descends, it is evident that the *velocity of P* must be the same as the *velocity of the wheel*.

2. Let the equal particles p, p', p'', p'''; o, o', o'', o'''; q, q', q'', q'''; &c, &c.; (placed in planes respectively parallel to the plane of the wheel A) be connected with each other and with the axis BC, in the same manner as the particles p, p', p'', p''' in the preceding Figure; but let the distances Sp, So, Sq, &c. at which they are placed from the axis, be *not equal to the radius of the wheel A;* then it is evident that the *velocities* of these particles will not be equal to the velocity of P, and, consequently, that their *inertia* (as opposed to the descent of P) cannot be measured by the quantities of matter contained in them. Before we can ascertain what their inertia is with respect to P, we must find what quantity of matter, placed *at the distance of the radius of the wheel* (A) from the axis of motion, will have the *same inertia* as the particles p, p', p'', p'''; o, o', o'', o'''; q, q', q'', q'''; &c. &c. placed at the distances Sp, So, Sq, &c. Let $p+p'+p''+p'''=u$; $o+o'+o''+o'''=u'$; $q+q'+q''+q'''=u''$, &c.; and let the *radius of the wheel* $=r$, then it will be found that a quantity of the matter equal to $\frac{u \times Sp^2}{r^2}$, placed at the distance (r) from the axis, will have the same inertia as (u) placed at the distance



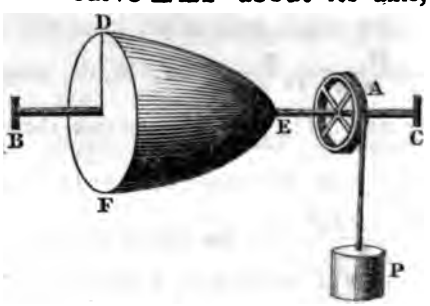
distance Sp[a]; that $\frac{u' \times So^2}{r^2}$, placed at the distance (r), will have the same inertia as (u') placed at the distance So; &c, &c. so that the *sum of the inertia* of all

([a]) Let M=the *momentum or* moving force of a body, Q= its *quantity of matter*, and V the *velocity* with which it moves, then $M \propto Q \times V$, and $Q \propto \frac{M}{V}$; suppose therefore that a particle of matter (p), placed at the distance Sp from the axis passing at right angles through S, be put in motion round S by the *given weight* P, and that the effect of P acting at $A=M$, at $p=m$; then the *quantity of matter* which could be put in motion by P *acting at* A (x) : *quantity of matter* which could be put in motion by P *acting at* p (p) :: $\frac{M}{Vy.\text{ at } A} : \frac{m}{Vy.\text{ at } p}$; now, by the property of the *lever* $M : m$:: $Sp : SA$:: $\frac{1}{SA} : \frac{1}{Sp}$; and from the nature of *angular motion*, Vely. at A : Vely. at p :: $SA : Sp$; hence $x : p :: \frac{1}{SA^2} : \frac{1}{Sp^2}$, and $x = \frac{p \times Sp^2}{SA^2}$; i. e. the effect is the same with respect to the moving force of P acting at A, whether the particle (p) revolve at the distance Sp, or a particle of matter equal to $\frac{p \times Sp^2}{SA^2}$ be placed at A, and acted upon by the same force. By applying this theorem to the case before us, it appears that the retarding force arising from the inertia of the particles $p+p'+p''+p'''$ (u); $o+o'+o''+o'''$ (u'); &c, &c. is the same with respect to the descent of P, whether u, u', u'', &c. be placed at their original distances Sp, So, Sq, &c, or a quantity of matter equal to $\frac{u \times Sp^2}{r^2} + \frac{u' \times So^2}{r^2} + \frac{u'' \times Sq^2}{r^2}$ + &c. be placed at the distance (r) from the axis, and acted upon by the same given force.

all these particles of matter may be represented by $\frac{u \times Sp^2}{r^2} + \frac{u' \times So^2}{r^2} + \frac{u'' \times Sq^2}{r^2}$ + &c. acting at the distance (r); in this case therefore the *accelerative force of P* will be measured by $\frac{P}{P + \frac{u \times Sp^2}{r^2} + \frac{u' \times So^2}{r^2} + \frac{u'' \times Sq^2}{r^2} + \&c.}$

$$= \frac{Pr^2}{Pr^2 + u \times Sp^2 + u' \times So^2 + u'' \times Sq^2 + \&c.}$$

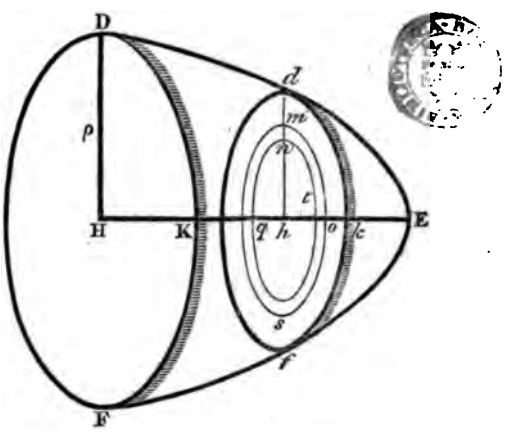
3. Suppose now the number of these particles p, p', p'', p'''; o, o', o'', o'''; q, q', q'', q'''; &c. &c. to be increased *ad infinitum*, so as to form a *solid DEF* of uniform density, generated by the revolution of an algebraic curve *DEF* about its axis, the *axis of the curve* coinciding with the *axis of rotation*; then if the *abscissa* of this curve $= x$, its corresponding *ordinate* $= y$, and its *greatest* ordinate $= \rho$, the sum of all the $u \times Sp^2 + u' \times So^2 + u'' \times Sq^2$ + &c. contained in this solid, will be equal to the fluent of $\frac{1}{2} \pi y^4 \dot{x}$ [a] (where $\pi = 3.141$ &c.) when



(a) Let *DEF* represent the solid, *EH* its *axis* (which coincides with the *axis of rotation*); let *dkf* be a circular section of it parallel to the base *DKF*; draw the ordinate *dh*, and let $Eh = x$, $dh = y$; with center *h* describe two concentric circles *mos*, *nqt*, indefinitely near to each other, and let the radius $hn = r$; then the

when $y=\rho$; hence *the force which accelerates the descent of* P (acting upon the wheel whose radius is

the circumference of the circle $nqt=2\pi v$, and the *annulus* $mosnqt=2\pi v\dot{v}$; hence "*the sum arising from multiplying each* "*particle of this annu-* "*lus into the square* "*of its distance from* "*the axis*" is equal to $2\pi v\dot{v}\times v^2=2\pi v^3\dot{v}$, and the *moment of inertia* for the circular plane $nqt=\int 2\pi v^3\dot{v}=\frac{1}{2}\pi v^4$; let $v=y$, then for the circular plane dkf it will be equal to $\frac{1}{2}\pi y^4$. For a small *cylinder* whose thickness is $\dot{x}$ it will be $\frac{1}{2}\pi y^4\dot{x}$, and consequently for the *part def* of the solid it will be $\int\frac{1}{2}\pi y^4\dot{x}$; and when estimated for the *whole solid DEF*, it will be the fluent of $\frac{1}{2}\pi y^4\dot{x}$ *when* $y=DH=\rho$. This expression is very readily applied to finding the *moment of inertia* of *any* algebraic solid revolving round its axis, but our present purpose will be answered by applying it to the *cylinder* and the *sphere*.



I. In the *cylinder* $y=\rho$; $\therefore\int\frac{1}{2}\pi y^4\dot{x}$ for a cylinder whose *length* is l, is $\int\frac{1}{2}\pi\rho^4\dot{x}$ when $x=l$, or $\frac{1}{2}\pi l\rho^4$; but the *solid content* of the cylinder$=\pi l\rho^2$; let $\pi l\rho^2=w=$*weight* of the cylinder, then $\frac{1}{2}\pi l\rho^2\times\rho^2$, or $\frac{1}{2}w\rho^2$, expresses the *moment of inertia*.

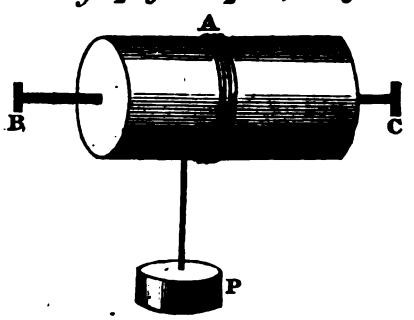
II. In the *sphere* $y^2=2\rho x-x^2$, $\therefore\int\frac{1}{2}\pi y^4\dot{x}=\int\frac{1}{2}\pi\dot{x}\times\overline{2\rho x-x^2}^2$ $=\int 2\pi\rho^2x^2\dot{x}-2\pi\rho x^3\dot{x}+\frac{1}{2}\pi x^4\dot{x}=\frac{2}{3}\pi\rho^2x^3-\frac{1}{2}\pi\rho x^4+\frac{1}{10}\pi x^5=$(when $x=2\rho$) $\frac{8\pi\rho^5}{15}$ for the *whole sphere*; but the solid content or *weight* (w) of the sphere$=\frac{4\pi\rho^3}{3}$, $\therefore$ the *moment of inertia*$=$ $\frac{2}{5}\times\frac{4\pi\rho^3}{3}\times\rho^2=\frac{2}{5}w\rho^2$.

is (r), whilst it communicates a rotatory motion to the solid DEF about the axis BC) is equal to $\frac{Pr^2}{Pr^2+f.\frac{1}{2}\pi y^4\dot{x} \text{ when } y=\rho}$. The quantity designated by $f.\frac{1}{2}\pi y^4\dot{x}$ is called the *moment of inertia*, because it expresses the *whole inertia of the revolving solid* as opposed to a force acting at a given distance from the axis of rotation.

EXAM. I.

A cylinder which weighs 100 *lbs* (w), *is put in motion by a weight of* 15 *lbs, attached to a string wound round it in the manner represented in the annexed Figure; It is required to find how far the weight* (P) *will descend from rest in* 3″.

Since, for the *cylinder* $f.\frac{1}{2}\pi y^4\dot{x}=\frac{1}{2}w\rho^2$, *the force which accelerates* P (F) = (by Art. 3) $\frac{Pr^2}{Pr^2+\frac{1}{2}w\rho^2}$; but in the present instance $\rho=r$, $\therefore F=\frac{P}{P+\frac{1}{2}w}=\frac{15}{15+50}=\frac{3}{13}$; consequently the space described by P in $3''=(mFT^2=^{(a)})\ 16\frac{1}{12}\times\frac{3}{13}\times 9=33.39$ feet.



Cor.

(a) For all quantities which may hereafter occur, expressive of the relation between S, F, V, T, m, the reader is referred to Art. 2, page 3, of this volume.

Cor. Let $P=w$, then $F=\frac{P}{P+\frac{1}{2}P}=\frac{2}{3}$; when a cylinder therefore is put in motion by a weight which is just equal to its *own* weight, then this latter weight descends with a force equivalent to $\frac{2}{3}ds$ *of its natural gravity.*

EXAM. II.

Suppose the weight (P) *in the preceding Example to be* 30*lbs, and that it descends through* 48 *feet in* 2″; *What is the weight of the cylinder?*

Let $x=$ weight of the cylinder, then, by Exam. 1, the *accelerative force of* P $(F)=\frac{P}{P+\frac{1}{2}x}=\frac{30}{30+\frac{1}{2}x}$; now $T^2=\frac{S}{mF}$, $\therefore$ (since $T=2$, $S=48$) we have $4=\frac{48}{16\frac{1}{12}}\times\frac{30+\frac{1}{2}x}{30}$, which gives $x=20\frac{5}{12}$ *lbs* for the *weight of the cylinder.*

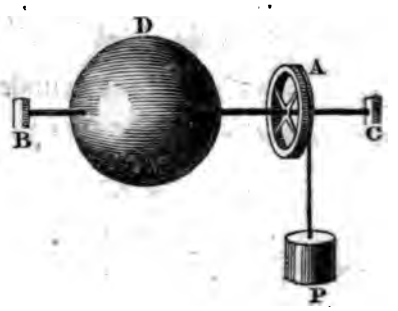
Cor. If the *weight of the cylinder* (w) be given, and it be required to find what weight will put it in motion, and at the same time descend through (S) feet in T''; then let $x=$ the weight required, and we have $F=\frac{x}{x+\frac{1}{2}w}$; hence $T^2=\frac{S}{mF}=\frac{S}{m}\times\frac{x+\frac{1}{2}w}{x}$, $\therefore x=\frac{\frac{1}{2}Sw}{mT^2-S}$.

EXAM. III.

A sphere D, whose radius is 3 *feet, and weight* 500*lbs, is put in motion by a weight of* 20*lbs, acting by means of*

of a string going over a wheel whose radius is 6 *inches; How long will the weight* (P) *be in descending through* 50 *feet, and what velocity will it have acquired at the end of its descent?*

In the *sphere* $\int \frac{1}{2}\pi y^4 \dot{x} = \frac{2}{5} w\rho^2$, $\therefore F = \frac{Pr^2}{Pr^2 + \frac{2}{5}w\rho^2} =$ (for $r = \frac{1}{2}$, $\rho = 3$) $\frac{20 \times \frac{1}{4}}{20 \times \frac{1}{4} + \frac{2}{5} \times 500 \times 9} = \frac{1}{361}$; hence the *time* in which P will descend through 50 feet $= \left(\sqrt{\frac{S}{mF}} = \right) \sqrt{\frac{50 \times 361}{16\frac{1}{12}}} =$ 33.5 seconds; and the velocity which it will have acquired at the end of that time $=$ $(2mFT =)$ $32\frac{1}{6} \times \frac{1}{361} \times 33.5 = 2.98$ feet in $1''$.



Cor. 1. Let $r = \rho$, then $F = \frac{P}{P + \frac{2}{5}w}$; if therefore the string be wound round a *great circle* of the sphere, in the same manner as it is wound round the *cylinder* in the preceding example, the force which will accelerate P in its descent will be $\frac{P}{P + \frac{2}{5}w}$ or $\frac{20}{20 + 200} = \frac{1}{11}$th of the force of gravity.

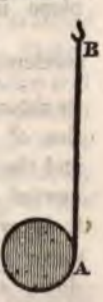
Cor. 2.

Cor. 2. If $P=w$, then $F=\frac{P}{P+\frac{2}{5}P}=\frac{5}{7}P$; if therefore the *descending* sphere in Cor. 1. be *equal* to the *revolving* sphere, the former will be acted upon by a force equivalent to $\frac{5}{7}$ *ths of its natural gravity* [a].

We

(a) From the *corollaries* to Examples 1 and 3, two inferences of considerable importance may be drawn; one of which respects the rectilinear descent of spheres and cylinders *by the unwinding of a string*; the other has reference to their *rolling or sliding* down inclined planes. We shall consider each case separately.

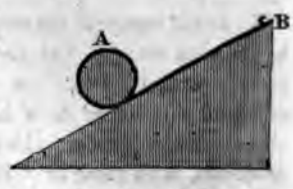
I. Let *A* represent the *section* of a cylinder or sphere descending by its own weight, and receiving at the same time a rotatory motion round its axis (or diameter) by the unwinding of the string *BA* fastened to a hook at *B*. Compare the circumstances under which *this* cylinder or sphere descends, with those of "the cylinder and sphere de-"scending by the string *AP* (and at the same time "communicating a rotatory motion to an *equal* cylinder "or sphere round the axis *BC*) in Cor. to Exam. 1, and "Cor. 2. to Exam. 3." The *moving force* in each case is the same, viz. "*the weight of the given cylinder or sphere*;" the *quantity of matter moved* is the same, for in each case it is "*the given cylinder or sphere + the moment of inertia arising from its rotatory motion*;" consequently the *accelerative force* must be the same. When a cylinder or sphere, therefore, thus descends by the unwinding of a string wrapt round it and fastened at *B*, the center of gravity of the *cylinder* will descend with an accelerative force equal to *two thirds*, and that of the *sphere* with an accelerative force equal to *five sevenths*, of the force of gravity. Let L=the *length* of the string, then for the *cylinder*, the *time of its unwinding* $=\left(\sqrt{\frac{S}{mF}}=\right)\sqrt{\frac{3L}{2m}}$ seconds; and the *velocity* acquired $=(\sqrt{4mFs}=)\sqrt{\frac{8}{3}mL}$ feet in 1″. For the *sphere*, $T=\sqrt{\frac{7L}{5m}}$; $V=\sqrt{\frac{20}{7}mL}$.



We have thus shewn the method of ascertaining the accelerative force of the descending body (P) by calculating the "*moment of inertia*" of the revolving body; let us next shew how this force may be estimated by means of its *center of gyration.*

LVI.

II. Suppose now the sphere or cylinder (A) to descend (by the *unwinding of the string* as before) down an *inclined plane*, then its *moving force* will be diminished in the ratio of the *height* of the plane (H) : *length* of the plane (L), whilst the *quantity of matter moved* remains the same; the *accelerative force* therefore will be diminished in the ratio of $H : L$. Hence the force which accelerates the *cylinder* down the inclined plane is $\frac{2}{3} \times \frac{H}{L}$, and that which accelerates the *sphere* is $\frac{5}{7} \times \frac{H}{L}$, of its natural gravity. Let the string be removed; then, if the elevation of the plane, and the adhesive force or friction between it and the body, be such as to create a force just equal to *the force exerted by the string in producing the rotatory motion of A*, it is evident that the body A would *roll* down the plane under precisely the same circumstances as when it descends down it *by the unwinding of the string BA*. Now the force with which it *slides* down the plane is $\frac{H}{L}$ *of its gravity*; hence the accelerative force with which a cylinder *rolls* down an inclined plane : the force with which it *slides* :: $\frac{2}{3} \times \frac{H}{L} : \frac{H}{L}$:: 2 : 3. For the same reason, in the *sphere*, the *rolling* force : *sliding* force :: $\frac{5}{7} \times \frac{H}{L} : \frac{H}{L}$:: 5 : 7.



This theorem for comparing the forces which accelerate cylinders or spheres according as they *roll* or *slide* down inclined planes, is only true on the condition that they have *precisely the same*

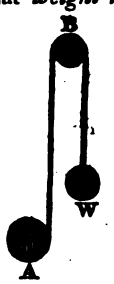
LVI.

On the mode of ascertaining the accelerative force of the descending body, when the inertia of the revolving body is estimated upon the principle of the center of gyration.

1. In the system of bodies p, p', p'', p'''; q, q', q'', q'''; &c. &c. (see Figure in page 50) the *moment of inertia* is $u \times Sp^2 + u' \times So^2 + u'' \times Sq^2 +$ &c. draw

same quantity of rotatory motion as when they descend by the unwinding of a string. To effect this, the elevation of the plane must be such as to give them a tendency to this degree of rotatory motion *at their first setting off*; and the adhesion or friction between them and the planes must be such as to *preserve* this tendency throughout their progress down the planes. Unless the elevation of the planes be thus nicely adjusted, the bodies will partly *roll* and partly *slide* down them; and if it were possible to render both the bodies and the planes *perfectly smooth* (so as to abolish the friction altogether), then would they *slide* down the planes under all circumstances of elevation; having in this case no tendency whatever to rotatory motion.

In considering the rectilinear descent of these bodies by the unwinding of the string, we supposed the string to be fastened to a *hook* at B; but if it were required to find what *weight* W, attached to the string and passing over a *pulley*, would be just sufficient to keep it motionless at B during the period of its unwinding, we have only to refer to the expressions for the accelerative force in Cor. to Exam. 1, and Cor. 2. Exam. 3, viz. $\frac{P}{P+\frac{1}{2}P}$ for the *cylinder*, and $\frac{P}{P+\frac{2}{5}P}$ for the *sphere*; the *denominators* of these fractions represent the *inertia* generated during the descent of the bodies, and, consequently, a weight (W) equal to $P+\frac{1}{2}P$

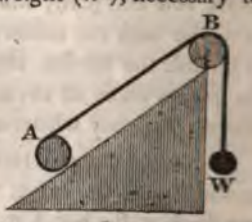


or

draw GR at right angles to the axis of rotation, and conceive the whole quantity of matter in the system to be *concentrated* in the point R; in that case the *moment of inertia* is $\overline{u+u'+u''+u'''+\&c.}\ GR^2$; hence, if we put $\overline{u+u'+u''+\&c.}\ GR^2 = u \times Sp^2 + u' \times So^2 + u'' \times Sq^2 + \&c.$ (in which case $GR = \sqrt{\frac{u \times Sp^2 + u' \times So^2 + u'' \times Sq^2 + \&c.}{u+u'+u''+\&c.}}\Big) R$ will be such a point, that, if the whole quantity of matter $u+u'+u''+\&c.$ were collected in it, "*the* "*moment of inertia would be the same as if that* "*matter*

or $\frac{3}{2}P$ for the *cylinder*, and $P+\frac{5}{9}P$ or $\frac{7}{5}P$ for the *sphere*, suspended from B, will be just sufficient to counteract this inertia.

Since the tendency of A to descend down an *inclined plane* is diminished in the ratio of $H : L$, the weight (W), necessary to keep the string ABW steady whilst (A) rolls down the plane, will also be diminished in that ratio, viz. $\frac{3P}{2} \times \frac{H}{L}$ for the *cylinder*, and $\frac{7P}{5} \times \frac{H}{L}$ for the *sphere*. To find therefore the relation between the *length* of the plane and the *height* of the plane, so that a given weight (W) suspended over the pulley (B) shall just keep the string motionless at B, whilst a given body (A) whose weight is (W) rolls down the plane; we have $W = \frac{3PH}{2L}$, or $L : H :: 3P : 2W$ for the *cylinder*; and $W = \frac{7PH}{5L}$, or $L : H :: 7P : 5W$ for the *sphere*. Thus, if $W = P$, then $L : H :: 3 : 2$ for the *cylinder*, and $L : H :: 7 : 5$ for the sphere; and if $W = \frac{1}{10}P$, $L : H :: 3 : \frac{2}{10} :: 15 : 1$ for the *cylinder*, and $L : H :: 7 : \frac{5}{10} :: 14 : 1$ for the *sphere*.



"*matter were spread through the system.*" This point R is called the *center of gyration*, and its distance (GR) from the axis of rotation may be easily calculated for any algebraic solid revolving round its own axis[a].

2. Having

([a]) It has already been shewn, that, in any algebraic solid, $u \times Sp^2 + u' \times So^2 + u'' \times Sq^2 + \&c. = f^{\cdot} \frac{1}{2}\pi y^4 \dot{x}$; and $u + u' + u'' + \&c.$ = *content of the solid*; hence

$$GR = \left(\sqrt{\frac{u \times Sp^2 + u' \times So^2 + u'' \times Sq^2 + \&c.}{u + u' + u'' + \&c.}} = \right) \sqrt{\frac{f^{\cdot} \frac{1}{2}\pi y^4 \dot{x}}{\text{Content of Solid}}}.$$

We shall apply this expression to finding the center of gyration of a *few* of these solids.

I. In the *cylinder* whose *radius* is (ρ) and *length* (l), $f^{\cdot}$ of $\frac{1}{2}\pi y^4 \dot{x}$ when $y = \rho$, is $\frac{1}{2}\pi l \rho^4$: *content of cylinder* $= \pi l \rho^2$, $\therefore GR = \sqrt{\frac{\frac{1}{2}\pi l \rho^4}{\pi l \rho^2}} = \rho\sqrt{\frac{1}{2}}$.

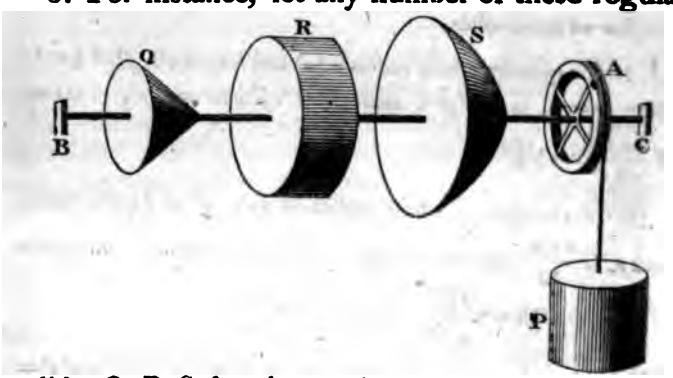
II. In the sphere whose radius is (ρ), $f^{\cdot}$ of $\frac{1}{2}\pi y^4 \dot{x}$, when $x = 2\rho$, is $\frac{8\pi\rho^5}{15}$ (see Note in page 53); the *content of the sphere* is $\frac{4\pi\rho^3}{3}$; $\therefore GR = \sqrt{\frac{8\pi\rho^5}{15} \times \frac{3}{4\pi\rho^3}} = \rho\sqrt{\frac{2}{5}}$.

III. In the *cone*, $y = ax$, $\therefore f^{\cdot} \frac{1}{2}\pi y^4 \dot{x} = f^{\cdot} \frac{1}{2}\pi a^4 x^4 \dot{x} = \frac{1}{10}\pi a^4 x^5 = \frac{1}{10}\pi x y^4 =$ (if the *height* of the cone $= h$, and *radius of its base* $= \rho$) $\frac{1}{10}\pi h \rho^4$; the *content of the cone* $= \frac{1}{3}\pi h \rho^2$; $\therefore GR = \sqrt{\frac{\frac{1}{10}\pi h \rho^4}{\frac{1}{3}\pi h \rho^2}} = \rho\sqrt{\frac{3}{10}}$.

IV. In the *paraboloid*, $y^2 = ax$, $\therefore f^{\cdot} \frac{1}{2}\pi y^4 \dot{x} = f^{\cdot} \frac{1}{2}\pi a^2 x^2 \dot{x} = \frac{1}{6}\pi a^2 x^3 = \frac{1}{6}\pi x y^4 =$ (when $x = h$, $y = \rho$) $\frac{1}{6}\pi h \rho^4$; the *content of the paraboloid* $= \frac{1}{2}\pi h \rho^2$; $\therefore GR = \sqrt{\frac{\frac{1}{6}\pi h \rho^4}{\frac{1}{2}\pi h \rho^2}} = \rho\sqrt{\frac{1}{3}}$.

2. Having found the value of GR in a solid of this kind, let it be put $=d$; and let $w=u+u'+u''+$ &c. $=$*weight of the solid*; then, from Art. 1, $wd^2 = u \times Sp^2 + u' \times So^2 + u'' \times Sq^2 +$ &c.; substitute this value for $u \times Sp^2 + u' \times So^2 + u'' \times Sq^2 +$ &c. in the expression of Art. 2, page 52, then *the accelerative force of* $P = \frac{Pr^2}{Pr^2 + wd^2}$(*). This expression for the accelerative force of P is more adapted to general purposes than that we have hitherto made use of $\left(\text{viz. } \frac{Pr^2}{Pr^2 + f^4 \frac{1}{2}\pi y^4 \dot{x}}\right)$.

3. For instance, let any number of these regular



solids, Q, R, S, &c. be put in motion by the weight (P), suspended by a string going over the wheel (A), whose

(*) In the *cylinder* $d=\rho\sqrt{\frac{1}{2}}$, $\therefore d^2=\frac{1}{2}\rho^2$, hence $F=\frac{Pr^2}{Pr^2+\frac{1}{2}w\rho^2}$; and in the *sphere* $d=\rho\sqrt{\frac{2}{5}}$, $\therefore d^2=\frac{2}{5}\rho^2$, consequently $F=\frac{Pr^2}{Pr^2+\frac{2}{5}w\rho^2}$; the expression for the accelerative force of P therefore, thus investigated, is the same as that in Exam. 1 and 3 of Section LV.

whose radius is (r); and let their *weights* be respectively denoted by w, w', w'', &c. and the *distances of their centers of gyration from the axis of rotation* by d, d', d'', &c.; then the *whole moment of inertia* arising from these revolving bodies will be measured by $wd^2 + w'd'^2 + w''d''^2 +$ &c. and consequently *the accelerative force of* P will be $\frac{Pr^2}{Pr^2 + wd^2 + w'd'^2 + w''d''^2 + \text{&c.}}$

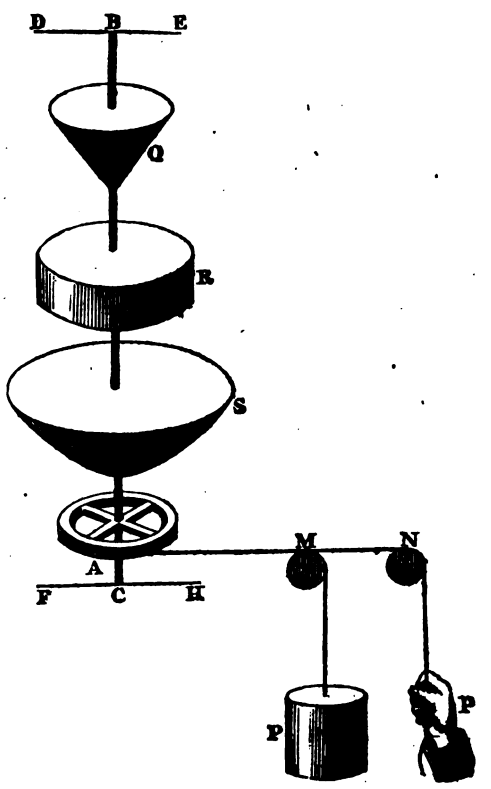
4. But supposing the matter of the bodies Q, R, S, &c. to be collected in their respective centers of gyration, then, by Art. 1, the distance of their *common center of gyration* from the axis will be $\sqrt{\frac{wd^2 + w'd'^2 + w''d''^2 + \text{&c.}}{w + w' + w'' + \text{&c.}}}$; let this distance $= D$, and let $w + w' + w'' + \text{&c.} = W$, then $D = \sqrt{\frac{wd^2 + w'd'^2 + w''d''^2 + \text{&c.}}{W}}$, and, consequently, $wd^2 + w'd'^2 + w''d''^2 + \text{&c.} = WD^2$; substitute this value for $wd^2 + w'd'^2 + w''d''^2 +$ &c. in the preceding article, then the *accelerative force of* $P = \frac{Pr^2}{Pr^2 + WD^2}$. We are come therefore to this general conclusion, viz. "That, if a rotatory motion be communi- "cated to a body or *system* of bodies (whose *weight* "*is* W), by the rectilinear descent of a weight (P) "acting at the distance (r) from the axis of rotation "passing through the respective *centers of gravity* "of the body or system; and, if D be the distance "of the *center of gyration* of the body or system "from that axis; then the *force which accelerates* "*the weight* (P) thus descending, is that part of the "force

"force of gravity which is expressed by the fraction "$\frac{Pr^2}{Pr^2+WD^2}$."

5. *Before* the weight begins to descend, *the pressure on the axis* BC is measured by $P+W$, being the whole weight with which it is loaded, independent of the weight of the wheel (A) and string (AP), which are supposed to be too small to be taken into the consideration. *After* it begins to descend, the axis will evidently be *relieved* from that part of the weight of P which is employed in its acceleration. If therefore the accelerative force of P in the preceding article be called F, then the part of its weight, of which *the axis is relieved*, will be expressed by FP; so that *the pressure upon the axis* BC *during the descent of* P will be measured by $P+W-FP$.

6. We have hitherto considered the body or *system* of bodies as revolving round an *horizontal* axis; but it may easily be shewn, that the foregoing expression for the *accelerative force of* P is applicable to the case when the bodies are made to revolve round a *vertical* one. Let DE, FH be two beams fixed parallel to the horizon, into which the moveable axis BC is inserted in a vertical direction; let the relative position of the bodies Q, R, S, &c. and the wheel (A) be the same as before; but suppose a small *pulley* fixed at M, whose upper edge just coincides with the plane of the wheel; then,

then, if the weight P descends by means of the string AMP going over the pulley at M, it is evident that the *moving force*, as well as the *inertia of the mass moved* (the inertia of the *pulley* M not being taken into consideration), is the same as before; consequently, the *accelerative force of* P will be the same; retaining therefore the notation of Art. 4, we may say that *in this case also* this force is measured by $\frac{Pr^2}{Pr^2+WD^2}$ (a).



7. If

(a) Supposing the beam FH and the pulley M to be fixed to the *same frame*, then it is evident, that *before* P begins to descend, the *pressure* upon this frame is measured by $P+W$, and that *during its descent* it is measured by $P+W-FP$, according to the principles laid down in Art. 5.

7. If the system be put in motion by a [a] *power* (P) acting over the pulley N (or by the same power acting immediately upon the wheel A when the axis is *horizontal*) instead of the *weight* (P), then the *inertia* of P is equal to nothing, $\therefore F = \frac{Pr^2}{Wd^2}$. The value of F may also be calculated thus; the *moving force* is (P); *the quantity of matter moved* is such "as, "when placed at the distance (r) from the axis of "rotation, shall produce the same effect, with re- "spect to P, as W placed in the *center of gyration*;" now, by Art. 2, page 50, that quantity of matter is $\frac{WD^2}{r^2}$; hence the *accelerative* force of (P) is $P \times \frac{r^2}{WD^2} = \frac{Pr^2}{WD^2}$

8. With respect to the *velocity* of the revolving system; since every point of the wheel (A) moves with the same velocity as the *weight* or *power* P, and since the velocity of P is *uniformly accelerated* (for this velocity is generated by the action of a *constant force*), the velocity of the *wheel* (A), and, consequently, that of the *whole revolving system*, will be uniformly accelerated. In estimating the velocity therefore of any point in the revolving body or system of bodies, after P has descended for any *time*

([a]) Whenever a body or system of bodies is said to be put in motion by a *power* as distinguished from a *weight*, we mean by it to express the action of *force without inertia*; such as the *exertion of animal strength*, &c. &c.

time (t), we in course mean the velocity with which the body or system would continue to revolve *uniformly*, were the action of P to *cease* as soon as it had descended for that time. We shall first find the *linear* velocity of any point in the revolving system, and then shew the method of ascertaining its *angular* velocity.

9. Let F = the *accelerative force* of P, and $m = 16\frac{1}{12}$ feet; then the *velocity* acquired by P after the *time* (t) is elapsed $= 2mFt$. But this is the *linear* velocity with which any point of the *wheel* (A) revolves at the end of that time; and it is evidently the linear velocity with which *any point of the body or system situated at the distance* (r) *from the axis* revolves[b]. Now the velocity at the distance r ($2mFt$) : the velocity at *any other distance* (x) $:: r : x$; hence the linear velocity of any point of the revolving system *situated at the distance* (x) *from the axis of rotation* $= \frac{2mFtx}{r}$; thus the linear velocity of the *center of gyration* at the end of the time (t) $= \frac{2mFtD}{r}$. If it were required to find the linear velocity of any point situated at the distance (x) from the axis, *after P had descended through any space* (s), then the velocity of P (i. e. the velocity at the

(b) For since the linear velocities in different circles described in the *same time* vary as the circumferences (i. e. as the *radii*) of the circles, the linear velocities of such points of the system as are at *equal distances* from the axis of rotation will of course be equal.

the distance r from the axis) $= \sqrt{4mFs}$ or $2\sqrt{mFs}$, and this velocity $2\sqrt{mFs}$ (V) : velocity at the distance (x) :: $r : x$, $\therefore V = \frac{2x\sqrt{mFs}}{r}$; thus the velocity of the *center of gyration* $= \frac{2D\sqrt{mFs}}{r}$.

10. Let $\pi = 3.141$, &c. then $2\pi r$ = the circumference of a circle whose radius is r; and since each point of this circumference revolves with a velocity of $2mFt$ feet in 1″, we shall have $2mFt : 2\pi r$:: the arc or angle described in 1″ (A) : 360°, $\therefore A = \frac{360^\circ \times mFt}{\pi r}$; i. e. the angle which would be described in 1″(a), by the body or system revolving *uniformly* with the velocity acquired *whilst P descends for the time* (t), is equal to $\frac{mFt}{\pi r} \times 360^\circ$, or at the rate of $\frac{mFt}{\pi r}$ *revolutions* in 1″. In the same manner it may be shewn that the body or system would perform $\frac{\sqrt{mFs}}{\pi r}$ revolutions in 1″, when it moves *uniformly* with the velocity acquired by P whilst *it descends through the space* (s).

11. Since

(a) This angle is the measure of the *angular velocity* of the body or system of bodies; and since 360°, m, and π, are *given* quantities, it *varies* as $\frac{Ft}{r}$. If t be also given, then it varies as $\frac{F}{r}$; i. e. "the angular velocity generated in a given time in a body "or system of bodies revolving round a fixed axis, varies *directly* "as the force which generates it, and *inversely* as the distance at "which that force acts from the axis."

11. Since the force which accelerates any point of the system is proportional to the *velocity*[b] of that point, and since the velocity varies as the *distance from the axis of rotation*, having given the accelerative force at *any one point* we can find it at any *other* point of the system; thus, the force which accelerates any point at the distance (r) from the axis $\left(\frac{Pr^2}{Pr^2+WD^2}\right)$: that which accelerates a point at the distance (x) :: r : x, $\therefore$ *the force at the distance* $(x) = \frac{Prx}{Pr^2+WD^2}$; for instance, the force which accelerates the *center of gyration* $= \frac{PrD}{Pr^2+WD^2}$, and so for any other point in the system.

12. By Art. 9. the *velocity* with which any point at the distance (x) from the axis revolves is $\frac{2x\sqrt{mFS}}{r}$, the *square* of this velocity, therefore, is $\frac{x^2\times 4mFs}{r^2}$; hence the *squares* of the velocity with which the *constituent particles* of the body or system (viz. p,p',p'', &c.) revolve at the distances Sp, Sp', Sp'', &c. will be represented by $\frac{Sp^2\times 4mFs}{r^2}$, $\frac{Sp'^2\times 4mFs}{r^2}$, $\frac{Sp''^2\times 4mFs}{r^2}$, &c. and the expression for *the sum of each*

(b) For $V \propto F\times T$; and if T be given, $V\propto F$, and $F\propto V$.

each particle multiplied into the square of its velo

will be $\frac{p \times Sp^2 + p' \times Sp'^2 + p'' \times Sp''^2 + \&c. \times 4mFs}{r^2}$ (

but by Art. 4, $p \times Sp^2 + p' \times Sp'^2 + p'' \times Sp''^2 + \&c.$
$= WD^2$, $\therefore S = \frac{WD^2 \times 4mFs}{r^2}$. Now suppose
system to be put in motion by the *power* P, t
$F = \frac{Pr^2}{WD^2}$, $\therefore S = \frac{WD^2 \times 4ms}{r^2} \times \frac{Pr^2}{WD^2} = P \times 4m$
$P \times$ *the square of the velocity which it would acq*
by moving freely through the space (s); hence "
" *sum of the products* arising from multiplying e
" *particle* of the revolving system into the *squ*
" *of the velocity* generated whilst P descends throu
" the space (s), is equal to the *product* arising fr
" multiplying P into the square of the velo
" which *would be generated in it* by moving fr
" through the space (s) under the influence of
" force of gravity [a]."

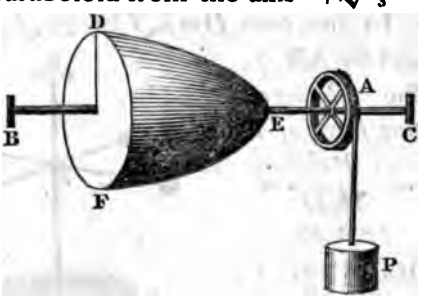
EXAM. I.

A paraboloid (DEF), *whose weight* (W) *is* 200
is put in motion by a weight (P) *of* 15 *lbs acting u*
the wheel A *whose radius* (r) 6 *inches; the radius*

([a]) It should be observed here, that this theorem is true o
when the system is put in motion by some *power*, i. e. by wei
without inertia; for suppose it to be put in motion by a *weight*
then $F = \frac{Pr^2}{Pr^2 + WD^2}$, $\therefore S$ is equal to $\frac{WD^2 \times 4ms}{r^2} \times \frac{Pr^2}{Pr^2 + W}$
or $\frac{PWD^2}{Pr^2 + WD^2} \times 4ms$, and *not to* $P \times 4ms$. *See the Note u*
this subject at the end of this Volume.

of the base of the paraboloid is 20 *inches; after the weight P has descended for* 10″, *it is taken off, and the paraboloid is left to revolve uniformly with the velocity acquired at that period; With what velocity does its center of gyration revolve, and how many revolutions does the paraboloid itself perform in a minute?*

By Note *a*, page 61, the distance (D) of the center of gyration of the paraboloid from the axis $= \rho\sqrt{\frac{1}{3}} = 20\sqrt{\frac{1}{3}}$; $\therefore$ the *accelerative force of* $P = \left(\frac{Pr^2}{Pr^2 + WD^2} = \right) \frac{15 \times 36}{15 \times 36 + 200 \times \frac{1}{3} \times 400} = \frac{1}{50.3} = F$. Now by Art. 9, the



velocity of the center of gyration $= \frac{2mFtD}{r} = \frac{32\frac{1}{6} \times 10 \times 20\sqrt{\frac{1}{3}}}{6 \times 50.3} = 12.1$ feet in 1″; and by Art. 10, the *number of revolutions which the paraboloid performs* in $1'' = \frac{mFt}{\pi r}$ [b] $= \frac{16\frac{1}{16} \times 10}{3.14 \text{ \&c.} \times \frac{1}{2} \times 50.3} = 2.03 =$ about 121 revolutions in a *minute*.

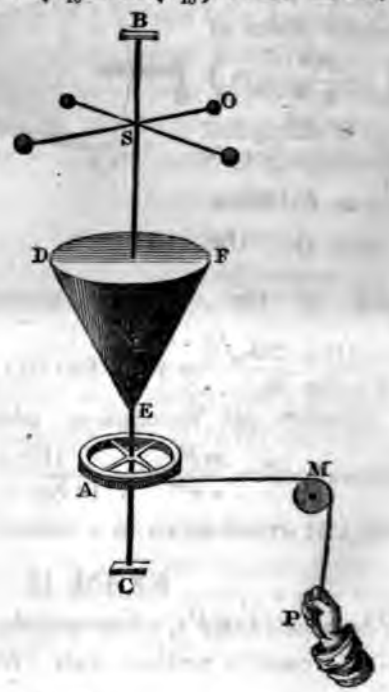
EXAM. II.

The cone (DEF), *whose weight is* 100 *lbs, is put in motion round a vertical axis* (BC) *by means of a power*

(b) Here $r = 6$ *inches* $= \frac{1}{2}$ a *foot*.

power (P) *of* 75 *lbs acting upon the wheel* (A), *whose radius* (r) *is* 5 *inches; a rotatory motion is at the same time communicated to a fly wheel* (O) *of inconsiderable weight, whose radius* (SO) *is* 3 *feet; the radius of the base of the cone* (ρ) *is* 25 *inches; after* P *has described a space of* 20 *feet, its action ceases; at that moment, what is the velocity of any point* (O) *of the fly wheel, and what number of revolutions does the machine itself perform in* 1″?

In the *cone* $D = \rho\sqrt{\frac{3}{10}} = 25\sqrt{\frac{3}{10}}$, $\therefore D^2 = 187.5$; and by Art. 7, the *accelerative force of* $P = \frac{Pr^2}{WD^2} = \frac{75 \times 25}{100 \times 187.5} = \frac{1}{10}$. Now by Art. 9, the velocity of any point at the distance (x) from the axis of rotation $= \frac{2x\sqrt{mFs}}{r}$, $\therefore$ the *velocity of the point* $O = \frac{2SO \times \sqrt{mFs}}{r} = \frac{2 \times 36 \times \sqrt{16\frac{1}{12} \times \frac{1}{10} \times 20}}{5} = 81.64$ feet in 1″; and by



by Art. 10, the *number of revolutions performed by the system* in $1'' = \frac{\sqrt{mFs}}{\pi r} = \frac{5.67}{3.14 \&c. \times \frac{5}{12}}$ [a] $= 4.3$, or *rather more than four in* $1''$.

EXAM. III.

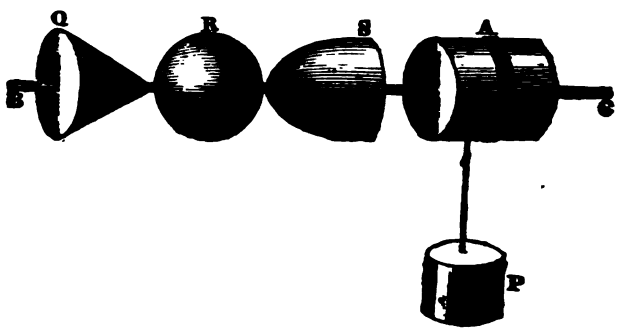
A solid, QRS (of the same form with that in page 143, *Part* I.) *whose weight is* 54 *lbs and radius one foot, is put in motion by a weight* (P) *of* 50 *lbs, by means of a string wound round the cylinder* (A), *whose dimensions are such as to circumscribe either of the parts Q, R, S of the solid; It is required to find the time* (t) *for which P must descend so as to produce a rotatory motion in the solid of* 10 *revolutions in* $1''$, *and also to determine the pressure upon the axis during its descent.* (*Fig. in page* 74.)

The distance of the *center of gyration* of this system from the axis is $\rho\sqrt{\frac{31}{75}}$, (D) [b], ρ being the

(a) In the quantity $\sqrt{\frac{mFs}{\pi r}}$, r must be expressed in *feet* to correspond with m and s, $\therefore r = 5$ *inches* $= \frac{5}{12}$ *feet*.

(b) By referring to page 143, Vol. I. it will be found that $Q=12$, $R=24$, $S=18$, $A=36$, and the weight (W) of the system $=90$ *lbs*, if d, d', d'', d''' be the *distances of the centers of gyration* of Q, R, S, A from the axis, and w, w', w'', w''' be their *weights*, then, by Art. 4, p. 63, $D = \sqrt{\frac{wd^2 + w'd'^2 + w''d''^2 + w'''d'''^2}{W}} =$ (for $w=12$, $w'=24$, $w''=18$, $w'''=36$, $d^2 = \frac{3}{10}\rho^2$, $d'^2 = \frac{2}{3}\rho^2$, $d''^2 = \frac{1}{3}\rho^2$, $d'''^2 = \frac{1}{2}\rho^2$, and $W=90$) $\rho\sqrt{\frac{31}{75}}$.

the radius of the cylinder; the *accelerative force of* P therefore $= \left(\frac{Pr^2}{Pr^2+WD^2}=\right) \frac{Pr^2}{Pr^2+\frac{31}{75}W\rho^2} =$ (since $\rho=r$) $\frac{P}{P+\frac{31}{75}W} = \frac{50}{50+\frac{31}{75}\times 90} = \frac{125}{218}$; the number of revolutions in 1″ which the system would perform with



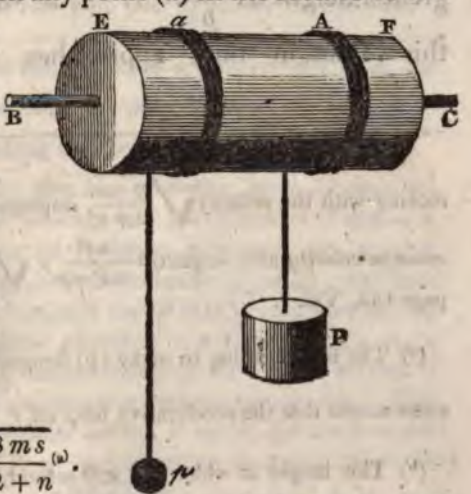
the velocity communicated to it whilst W descends for the time (t) is denoted by $\frac{mFt}{\pi r}$, which, in the present instance, is to be equal to 10; hence $\frac{mFt}{\pi r} = 10$, or $t = \frac{10\pi r}{mF} =$ (for $\pi = 3.14$ &c, $r=1$, $m = 16\frac{1}{129}$, $F = \frac{125}{218}$) 3.4 seconds. By Art. 5, page 64, the *pressure upon the axis* during the descent of $P = P + W - FP = 50 + 90 - \frac{125}{218}\times 50 = 111.34$ *lbs.*

EXAM. IV.

The weight of the cylinder EF (W) : P :: n : 1; *when* P *has descended from rest through the space* (s), *it*

it is taken off, and (p) *is attached to the string* (ap), *so as to be drawn up by the momentum which the cylinder has acquired; What is the greatest height to which* (p) *may be made to ascend?*

By Exam. 1, page 54, the accelerative force with which P descends is $\frac{P}{P+\frac{1}{2}W} = \frac{P}{P+\frac{1}{2}nP} = \frac{2}{2+n}$; $\therefore$ the *velocity* which it acquires in falling through the space $(s) = (\sqrt{4mFs} =) \sqrt{\frac{8ms}{2+n}}$, and this is the velocity with which any point (a) in the surface of the cylinder moves *at the instant* P *is taken off.* Just at that instant *the weight* (p) *is attached to the string* (ap), and the cylinder and (p) begin to move with the *common velocity* $\frac{nP}{nP+p} \times \sqrt{\frac{8ms}{2+n}}$ [a].



But this common velocity is *the velocity with which*

(a) For if A, moving with velocity (a), impinges on B at *rest*, then the *common velocity* after impact $= \frac{Aa}{A+B}$; if therefore nP, moving

which (p) *begins to ascend against the action of the force* $\frac{p}{p+\frac{1}{2}nP}$ or $\frac{2p}{2p+nP}$[a]; the height therefore *to which* (p) *may ascend*[b] $= \left(\text{for } S = \frac{V^2}{4mF}\right)$

$$\frac{n^2P^2}{\overline{nP+p}^2} \times \frac{8ms}{2+n} \times \frac{2p+nP}{8mp'} = \frac{n^2P.\overline{2p+nP}}{\overline{nP+p}^2.\overline{2+n}} \times \frac{Ps}{p}.$$

Cor. 1. Since $\frac{n^2P.\overline{2p+nP}}{\overline{nP+p}^2.\overline{2+n}}$ is always a *proper fraction*[c], the weight (p) can never ascend to so great a height as $\frac{Ps}{p}$. If n be *indefinitely great*, this fraction only approaches to *unity* as its limit,

moving with the velocity $\sqrt{\frac{8ms}{2+n}}$, impinges on (p) at rest, their *common velocity* after impact $= \frac{nP}{nP+p} \times \sqrt{\frac{8ms}{2+n}}$. See Art. 6, page 158, Vol. I.

(a) The force tending to make (p) descend is $\frac{p}{p+\frac{1}{2}nP}$, for the same reason that the accelerative force of P is $\frac{P}{P+\frac{1}{2}nP}$.

(b) The height to which (p) will ascend, is the space through which it must fall under the action of the force $\frac{p}{p+\frac{1}{2}nP}$ to acquire the velocity $\frac{nP}{nP+p} \times \sqrt{\frac{8ms}{2+n}}$.

(c) For the *numerator* is $n^2P^2+2n^2Pp$, and the *denominator* $n^2P^2+2n^2Pp+(2n^2P^2+4nPp+\overline{n+2}.p^2)$.

limit[d]; even in this case, therefore, the height ascended through by (p) cannot *exceed* $\frac{Ps}{p}$[e].

Cor. 2. If $n=1$, the space ascended through by $(p)=\frac{P.\overline{2p+P}}{3.\overline{P+p}|^2}\times\frac{Ps}{p}$; and if at the same time $p=P$, this space is equal to $\frac{1}{4}s$; i.e. "if the cylin- "der be put in motion by a weight equal to its "*own* weight, then the greatest height to which it "can draw up *another equal weight* is only *one* "*fourth* part of the space through which the first "weight descended to put the cylinder in motion."

(d) The value of the fraction, when n is indefinitely great, approaches to $\frac{n^3P^2}{n^3P^2}$ as its limit, the other terms of the numerator and denominator vanishing with respect to n^3P^2.

(e) According to the *first law of motion*, after the cylinder has acquired a certain velocity, and the action of P ceases, it would go on revolving *ad infinitum*, if it were not impeded by the *friction on its axis*, and the *resistance of the air*; but from what is here shewn, it appears that *if that friction were entirely abolished, and the motion performed in a perfect vacuum*, yet if it were exposed to the *constant action* of the smallest particle of matter, its revolutions would cease within a limited time. Thus suppose the cylinder to weigh 100 *lbs*, P *one lb*, and that p is only $\frac{1}{20}$ *th part of an ounce*, then, by substituting the proper values for n, P, p, in the expression $\frac{n^2P.\overline{2p+nP}}{\overline{nP+p}|^2.2+n}$, it will be found, that the motion would cease after p has ascended through a space equal to $\frac{100}{101}\times\frac{Ps}{p}$, or about 316 times s.

Lecture

Lecture XV.

ON

THE ASCENT AND DESCENT OF WEIGHTS, OVER PULLEYS AND WHEELS WHOSE INERTIA IS TAKEN INTO CONSIDERATION.

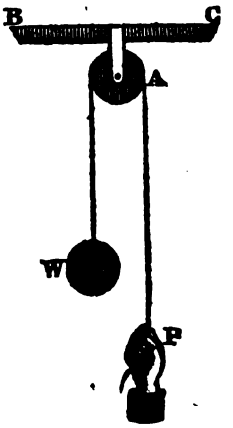
WE have hitherto considered only the case, when "*a given force communicates a rotatory motion to a* "*body or system of bodies;*" we are next to estimate the effects produced, when "*a given force is employ-* "*ed to move a given weight, by the intervention of* "*a body or system of bodies endued with a rotatory* "*motion,*" as when a weight is made to ascend by a force acting over a *pulley* or *wheel*, or a *combination* of pulleys or wheels.

LVII.

On the Motion of Bodies over Pulleys.

1. Let the weight (W) be raised by means of the *power* (P) acting over a single *fixed pulley* (A) whose *weight* is (w); then the *moving force* is $P - W$;

$P-W$; and the *matter moved*, W + *the inertia of the pulley* $(\frac{1}{2}w)$[a], P having *no* inertia; hence the *accelerative force of* $P=\frac{P-W}{W+\frac{1}{2}w}$; but since P and W move with the same velocity, this is *also* the accelerative force with which W ascends; $\therefore$ the *accelerative force of* $W=\frac{P-W}{W+\frac{1}{2}w}$[b].

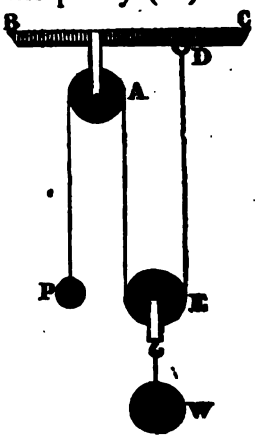


2. Let us next estimate the ascending force of W, when it is raised by a *weight* (P) acting over a *moveable* pulley (E) as well as a *fixed* one (A). In this case, the descent of P is opposed by as much of the weight of W as is supported *by the string* AE, which is evidently $\frac{1}{2}W$; the *moving force* therefore is $P-\frac{1}{2}W$. Supposing the pulleys to be *equal*, and the

(a) Let the pulley be considered as a small *cylinder*, whose *radius* is ρ; then the distance (D) of its *center of gyration* from the axis $=\rho\sqrt{\frac{1}{2}}$; now if (w) be the quantity of matter placed in the center of gyration, then the quantity to be placed in the circumference of the pulley (viz. at the distance from the axis where P acts) is $\frac{wD^2}{\rho^2}$; but $D^2=\frac{1}{2}\rho^2$, $\therefore \frac{wD^2}{\rho^2}=\frac{1}{2}w=$the *inertia of* w as opposed to the action of P.

(b) In this and the following section we shall estimate the *ascending force of* W, and not the *descending force of* P; as *the weight raised in a given time* is the proper measure of the effect produced by any given machine.

the weight of each to be (w); then, for the same reason as before, the inertia of the pulley (A) will be $\frac{1}{2}w$; but since the pulley (E) moves with only *half* the velocity of the pulley (A), *its* inertia will be only *one fourth*[a] of the inertia of (A), i. e. $\frac{1}{8}w$; the *whole inertia* therefore of the pulley (E), arising both from its *weight* and *rotatory motion*, will be $w+\frac{1}{8}w$ or $\frac{9}{8}w$. Again, since W moves with only *half* the velocity of P, its inertia, as opposed to P, will be only $\frac{1}{4}W$; hence the *whole mass moved* is $P+\frac{1}{4}W+\frac{1}{2}w+\frac{9}{8}w=P+\frac{1}{4}W+\frac{13}{8}w$; and since the *moving force* is $P-\frac{1}{2}W$, the *accelerative force of* P must be $\frac{P-\frac{1}{2}W}{P+\frac{1}{4}W+\frac{13}{8}w}=\frac{8P-4W}{8P+2W+13w}$; and the force *with which* W *ascends* is $\frac{4P-2W}{8P+2W+13w}$[b].



(a) Since the inertia of any particle of matter (p), placed at the distance (x) from the axis of rotation (and opposed to a force acting at the *given* distance (r) from that axis), is equal to $\frac{px^2}{r^2}$, this inertia must *vary* as x^2; but its velocity (v) varies as x, ∴ the *inertia* varies as the *square of the velocity*; it may therefore be considered as a fundamental principle as to the motion of a *particle of matter* when referred to a fixed axis of rotation (and consequently of a *body or system of bodies* when collected in their centers of *gravity* or *gyration*) that its *inertia* varies as the *square of the velocity* with which it moves.

(b) Since W moves with half the *velocity* of P, it can only be acted upon by half the *accelerative force* of P.

EXAM. I.

A weight (W) *of* 100 *lbs is raised through a space of* 40 *feet in* 10″, *by means of a power* (P) *acting over a fixt pulley whose weight* (w) *is* 2 *lbs.; What is the magnitude of the power, and what is the pressure upon the axis of the pulley during the time of its action?*

Let x = the magnitude of the power, then, by Art. 1, the *force with which W ascends* $\left(=\frac{P-W}{W+\frac{1}{2}w}\right)$ $=\frac{x-100}{101}=F$; now the space described in 10″ by a body moving by the action of the force $F=$ $(mFT^2=)$ $16\frac{1}{12}\times\frac{x-100}{101}\times 100$; which in the present instance is to be 40 feet, $\therefore 16\frac{1}{12}\times\overline{x-100}\times 100$ $=40\times 101$, or $x=102.51$ *lbs*. The *pressure upon the axis of the pulley* (See Note *a*, page 84) $=$ $\frac{4PW}{P+W}=\frac{4Wx}{x+W}=202.47$ *lbs*; and therefore the axis is relieved of only $\frac{1}{25}$th of a lb. from the *whole pressure* (202.51 *lbs*) arising from $P+W$.

EXAM. II.

What weight could be raised by another weight of 50 *lbs, acting by means of a string going round a fixt and moveable pulley* (*the weight of each of which is one lb.*) *through a space of* 30 *feet in* 6″?

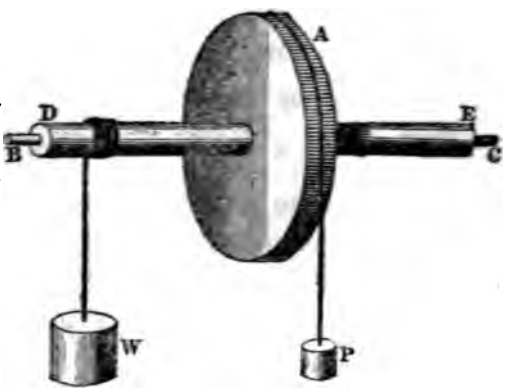
Let $x=$ the weight; then, by Art. 2, the *force with which x ascends* $=\frac{4P-2x}{8P+2x+13}=\frac{200-2x}{400+2x+13}=F$;

the space described under the influence of the force F in 6" $(mFT^2) = 16\frac{1}{12} \times \frac{200-2x}{400+2x+13} \times 36 =$ (by the Problem) 30; $\therefore x = 84.8$ *lbs.*

LVIII.

On the motion of bodies over a Wheel and Axle, or a combination of Wheels and Axles.

1. Let a weight P, acting upon the *wheel* (A) whose radius is (r), draw up the weight W by means of a string going over the *axle* DE whose radius is (ρ); and let the *weight* of the wheel and axle $= w$, the *distance of its center of gyration* from the axis $(BC) = d$[a]. By the common property



([a]) Let w' = weight of the *wheel*, w'' = weight of the *axle*; and let d', d'' be the respective distances of their *centers of gyration* from the axis of motion; then, by Art. 4, page 63, $d = \sqrt{\frac{w'd'^2 + w''d''^2}{w' + w''}} =$ (for $d' = r\sqrt{\frac{1}{2}}$, and $d'' = \rho\sqrt{\frac{1}{2}}$) $\sqrt{\frac{\frac{1}{2}w'r^2 + \frac{1}{2}w''\rho^2}{w}}$.

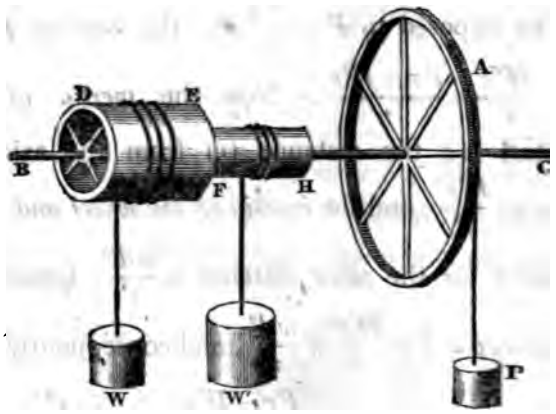
perty of the wheel and axle, the *descending tendency* of W as opposed to $P=\frac{W\rho}{r}$, $\therefore$ the *moving force* $=P-\frac{W\rho}{r}=\frac{Pr-W\rho}{r}$. Now the *inertia of W* estimated for the distance (r) from the axis of motion is $\frac{W\rho^2}{r^2}$, and the *inertia of the wheel and axle* estimated for the same distance is $\frac{wd^2}{r^2}$; hence the *mass moved* $=P+\frac{W\rho^2}{r^2}+\frac{wd^2}{r^2}$, and consequently the *accelerative force of* $P=\frac{Pr-W\rho}{r}\times\frac{r^2}{Pr^2+W\rho^2+wd^2}$ $=\frac{Pr^2-Wr\rho}{Pr^2+W\rho^2+wd^2}$; but by Art. 11, page 69, the accelerative force of W : that of $P :: \rho : r$, $\therefore$ the *force with which W ascends* $(F)=\frac{Pr\rho-W\rho^2}{Pr^2+W\rho^2+wd^2}$ [b].

2. Sup-

(b) If the *inertia of* $P=0$, then $F=\frac{Pr\rho-W\rho^2}{W\rho^2+wd^2}$; if the *inertia of* $w=0$, then $F=\frac{Pr\rho-W\rho^2}{Pr^2+W\rho^2}$; and if the inertia of *both P and w* $=0$, then $F=\frac{Pr\rho-W\rho^2}{W\rho^2}$.

By attentively considering all the circumstances under which the motion of this machine is performed, it will appear, that the *pressure upon the axis* (BC) during the descent of P is equal to $P+W+w-\overline{P+W}\times$ *the force which accelerates the descent of the common center of gravity of P and W.* Now, since P and W are on *different sides* of the axis BC, the *distance of their common center*

2. Suppose now that the weight P, acting



a *very light wheel* (A) whose radius is (r), dra

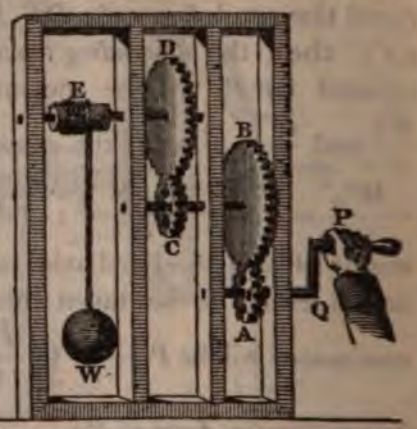
center of gravity from it $= \frac{Pr - W\rho}{P+W}$. But *the force which* ... *lerates* $P = \frac{\overline{Pr - W\rho} \times r}{Pr^2 + W\rho^2 + wd^2}$; $\therefore$ by Art. 11, page 69, the *which accelerates the common center of gravity of* P *and* $\frac{Pr - W\rho}{Pr^2 + W\rho^2 + wd^2} \times \frac{Pr - W\rho}{P+W}$; and consequently the *pressur* *the axis during the descent of* $P = P + W + w - \frac{\overline{Pr - W\rho}}{Pr^2 + W\rho^2 +}$ and if the weight of the wheel and axle be very inconsid (i. e. if $w=0$), then this pressure is equivalent to $\frac{PW.r+}{Pr^2+}$ Upon the same principle, the weight sustained by the axis ... *single fixed pulley* might be estimated; and as, in this case, this pressure (when the *weight of the pulley* is not take consideration) will be represented by $\frac{4PW}{P+W}$

the *two weights* W, W', by means of strings wound round the *two hollow axles* DE, FH, whose radii are ρ, ρ'; then the *descending tendencies* of W, W', as opposed to P, will be measured respectively by $\frac{W\rho}{r}$ and $\frac{W'\rho'}{r}$; hence the *moving force* is $P-\frac{W\rho}{r}-\frac{W'\rho'}{r}=\frac{Pr-W\rho-W'\rho'}{r}$; and as in this case the weight of the wheel and axles may be reckoned too inconsiderable to be taken into the account, the *mass moved* will be $P+\frac{W\rho^2}{r^2}+\frac{W'\rho'^2}{r^2}$; consequently, the *accelerative force of* $P=\frac{Pr^2-Wr\rho-W'r\rho'}{Pr^2+W\rho^2+W'\rho'^2}$ (ϕ); moreover, the *force with which* W *ascends* $=\left(\frac{\phi\rho}{r}=\right)\frac{Pr\rho-W\rho^2-W'\rho\rho'}{Pr^2+W\rho^2+W'\rho'^2}$; and the *force with which* W' *ascends* $=\left(\frac{\phi\rho'}{r}=\right)\frac{Pr\rho'-W\rho\rho'-W'\rho'^2}{Pr^2+W\rho^2+W'\rho'^2}$ [a].

3. In

(a) If there be n *weights* W, W', W'', &c. drawn up by the *power* P acting upon the *wheel* A, which turns n *axles* whose radii are ρ, ρ', ρ'', &c. then it might easily be shewn, that the *force with which* W *ascends* $=\frac{Pr\rho-W\rho^2-W\rho\rho'-W''\rho\rho''-\&c.\text{ to }\overline{n+1}\text{ terms}}{Pr^2+W\rho^2+W'\rho'^2+W''\rho''^2+\&c.\text{ to }\overline{n+1}\text{ terms}}$; that with which W' *ascends* $=\frac{Pr\rho'-W\rho\rho'-W'\rho'^2-W''\rho'\rho''-\&c.}{Pr^2+W\rho^2+W'\rho'^2+W''\rho''^2+\&c.}$; that with which W'' *ascends* $=\frac{Pr\rho''-W\rho\rho''-W'\rho'\rho''-W''\rho''^2-\&c.}{Pr^2+W\rho^2+W'\rho'^2+W''\rho''^2+\&c.}$; &c. &c. the *law* of which expressions is manifest.

3. In the annexed Figure, let the radii of the *wheels* D, B, and *handle* $PQ = r, r', r''$ respectively; the radii of the two *small wheels* C, A, and of the *axle* E, each $= 1$; the distance of the center of gyration of the *axle E and wheel D* (whose *weight* is w) from the axis of rotation $= d$, of *wheels* C and B (whose *weight* is w') $= d'$, of *wheel A and handle PQ* (whose *weight* is w'') $= d''$; and suppose the weight W to be drawn up round the *axle* E, by means of a *power* P acting upon the *handle PQ*. By Art. 7, page 258, Part II, the *descending tendency of* W as opposed to $P : W :: 1 : rr'r''$, $\therefore$ the *moving force* of $P = P - \frac{W}{rr'r''} = \frac{Prr'r'' - W}{rr'r''}$. By calculating the *inertia* of the different parts of the machine upon the principles already explained, the *whole mass moved* will be found to be $\frac{W + wd^2 + w'd'^2r^2 + w''d''^2r^2r'^2}{r^2r'^2r''^2}$ (a); hence the *accelerative force*



(a) In making this calculation, we shall begin with the weight (W), and trace the *increase of inertia* through each separate part of the machine, until we arrive at the power (P) acting at the extremity of the handle PQ. Now the inertia of the *weight* W and of the *wheel D and axle E*, as transferred to a point in the circumference

force of $P = \frac{Prr'r'' - W.rr'r''}{W + wd^2 + w'd'^2r^2 + w''d''^2r^2r'^2}(\phi)$; and the force *with which W ascends* $= \frac{\phi}{rr'r''}$ [b] $= \frac{Prr'r'' - W}{W + wd^2 + w'd'^2r^2 + w''d''^2r^2r'^2}$ [c].

EXAM. I.

A weight of 500 *lbs* (W) *is raised up by a rope wound round an axle whose radius is* 6 *inches*; *the weight of the wheel and axle is* 80 *lbs, and the distance of their center of gyration from the axis of rotation is* 3 *feet*; *It is required to find the radius and weight of the wheel, so that another weight* (P) *of* 100 *lbs, acting at its circumference, may make W ascend through a space of* 10 *feet in* 5″; *and the pressure upon the axis during the descent of P.*

Let

circumference of the wheel D, is $\frac{W + wd^2}{r^2}$ (M), and this is the inertia opposed to the motion of the *wheel C*; again, M *acting at the circumference of the wheel C* + the *inertia of the wheels C and B* estimated for a point in the circumference of the *wheel B* or $\frac{M + w'd'^2}{r'^2}$ $\left(= \frac{W + wd^2 + w'd'^2r^2}{r^2r'^2} = M'\right)$ is the inertia opposed to the motion of the *wheel A*; lastly, M' *acting at the circumference of the wheel A* + the *inertia arising from the wheel A and handle PQ* estimated for the distance PQ from the axis AQ or $\frac{M' + w'd''^2}{r''^2}$ $\left(= \frac{W + wd^2 + w'd'^2r^2 + w''d''^2r^2r'^2}{r^2r'^2r''^2}\right)$ is the *whole inertia* opposed to the action of the *power P*.

[b] For *velocity* of P : *velocity* of W :: W : P (when they are in æquilibrio) :: $rr'r''$: 1; ∴ *force* of P (ϕ) : *force* of W :: $rr'r''$: 1, or *force of* $W = \frac{\phi}{rr'r''}$.

[c] If $r = r' = r''$; $w = w' = w''$; and $d = d' = d''$; then *the force with which W ascends* $= \frac{Pr^3 - W}{W + wd^2 + wd^2r^2 + wd^2r^4}$ (F);

and

Let x = the *radius of the wheel*; then, referring to Art. 3, we have $P=100$, $W=500$, $\rho=\frac{1}{2}$, $w=80$, $d=3$, $\therefore$ the *force with which W ascends* $=\frac{50x-125}{100x^2+125+720}=F$; but $S=mFT^2$, or $10=\frac{16\frac{1}{16}\times 50x-125\times 25}{100x^2+845}$, which equation in *whole numbers* is $x^2-20x+58=0$, from which $x=10+\sqrt{42}=16.4$ feet for the *radius of the wheel*. To find the *weight* of the wheel, we must have recourse to the expression in Note (a), page 82, where $d=\sqrt{\frac{\frac{1}{2}w'r^2+\frac{1}{2}w''\rho^2}{w'+w''}}$; in the present instance $d=3$, $r=16.4$, $\rho=\frac{1}{2}$, $\therefore\ 3=\sqrt{\frac{134.4w'+\frac{1}{8}w''}{w'+w''}}$, from which it appears that w' (the weight of the *wheel*) : w'' (the weight of the *axle*) :: 1 : 14 very nearly[a]; and as the weight of the *wheel and axle* is 80 *lbs*, the weight of the *wheel* must be $\frac{80}{15}$ or

and if there be n such wheels and axles, it is easily shewn that $F=\frac{Pr^n-W}{W+wd^2+wd^2r^2+\&c....wd^2r^{2n-2}}=\frac{Pr^n-W}{W+wd^2.1+r^2+\&c....r^{2n-2}}=\frac{Pr_n-W}{W+wd^2.\frac{r^{2n}-1}{r^2-1}}$.

(a) For $9w'+9w''=134.4w'+\frac{1}{8}w''$, or $8\frac{1}{8}w''=125.4w'$; $\therefore 71w''=1003.2w'$, and $w' : w'' :: 71 : 1003.2 :: 1 : 14.12$.

or $5\frac{1}{3}$ *lbs*; consequently the weight of the *axle* is $74\frac{2}{3}$ *lbs*. With respect to the pressure upon the *axis of rotation during the descent of* P (p), we have

$\frac{\overline{Pr - W\rho}|^2}{Pr^2 + W\rho^2 + wd^2}$ (see Note *a*, page 84) $= 69.6$ *lbs*,

$\therefore\ p = P + W + w - 69.6 = 680 - 69.6 = 610.4$ *lbs*.

EXAM. II.

Three weights (W, W', W'') *which are to each other as* 1, 2, 3, *are drawn up by another weight* (P), *in the manner represented in Figure, page* 84; $P : W :: 10 : 1$; *the radii* ρ, ρ', ρ'' *of the axles about which* W, W', W'' *are respectively drawn up, are to each other as* 3, 2, 1; *and the radius of the wheel* A (r) $: \rho :: 2 : 1$; *It is required to find the spaces through which* W, W', W'' *are severally drawn up in* 1″.

Since $r : \rho\ (3) :: 2 : 1$, $r = 6$; and since $P : W :: 10 : 1$, $P = 10$. Substituting therefore the values here assigned to P, W, W', W''; and r, ρ, ρ', ρ''; in the expressions of Note (a), page 85; we have, the force *with which* W *ascends* $= \frac{15}{38}$ths, the force *with which* W' *ascends* $= \frac{10}{38}$ths, and the force *with which* W'' *ascends* $= \frac{5}{38}$ths, of the force of *gravity*. These forces, consequently, are as 3, 2, 1; but the *spaces* described in a *given time* are proportional to the *forces*; hence the *spaces ascended through, by*

W, W', W'' respectively, in 1″, are as 3, 2, 1, and are *equal* to $\frac{15}{38}m$, $\frac{10}{38}m$, $\frac{5}{38}m$ (where $m = 16\frac{1}{12}$ feet).

EXAM. III.

In the Figure of page 86, *the radii of the wheels D, B, and the handle PQ, are each equal to four times the radii of the wheels A, C, and axle E; the quantity of matter put in motion round each axis of rotation is* 10 *lbs, and the distance of its center of gyration from that axis is half the radius of the wheels D or B; It is required to find the magnitude of the power (P) which may be sufficient to elevate a ton weight (W) through the space of* 10 *feet in a minute.*

This is a case of Note (*c*), page 87, where $W = 2240$, $r = 4$, $n = 3$, $w = 10$, $d = 2$, $\therefore F = \frac{Pr^n - W}{W + wd^2 \cdot \frac{r^{2n}-1}{r^2-1}} = \frac{64P - 2240}{13160} = \left(\text{for } F = \frac{S}{mT^2}\right)$ $\frac{10}{16\frac{1}{12} \times 3600''}$; hence $64P - 2240 = 2.27$, and $P = 35.03$ *lbs.*

Lecture XVI.

ON

THE MAXIMUM EFFECTS OF MACHINERY.

In estimating the *maximum effects* of Machinery, we shall begin by ascertaining under what circumstances a given power must act "*to produce the* "*greatest angular velocity in a given body or system* "*of bodies revolving about a fixed axis.*"

LIX.

On the method of finding the distance from the axis of rotation at which a given force must be applied, so as to communicate the greatest angular velocity in a given time to a body or system of bodies revolving round that axis.

1. By Art. 10, p. 68, if $m=16\frac{1}{16}$ feet, $\pi=3.141$ &c, $F=$ the *accelerative force* of the system, $t=$ the *time* for which that force has acted, and $x=$ the *distance* from the axis of rotation (whether *horizontal* or *vertical*)

vertical) at which it acts; then $\frac{mFt}{\pi x}$ = the *number of revolutions* which the body or system would perform in 1″, if left to revolve uniformly with the velocity acquired at the end of that time; the *angular velocity* therefore of the system will be greatest when *this quantity* is greatest; but before we can find when $\frac{mFt}{\pi x}$ is a *maximum*, we must substitute the value of F.

2. Now suppose the body or system to be put in motion by a given *power* (P), that the weight of the system is (W), and that the *distance of its center of gyration.* from the axis is (D); then, by Art. 7, page 66, $F = \frac{Px^2}{WD^2}$, $\therefore \frac{mFt}{\pi x} = \frac{mtPx}{\pi WD^2}$, which (when t is given) varies as x. The *angular velocity* therefore produced by the action of a given *power* (P) admits of no *maximum*, but keeps continually increasing, as the distance at which it acts from the axis increases.

3. But the case is quite different when the system is put in motion by the action of a given *weight* (P); for since the *inertia* of the weight increases as the *square* of its distance from the axis of rotation, whilst the efficacy of its *mechanical action* increases only *as the distance*, it is evident that (by increasing this distance) its effect to produce a rotatory motion in the system *will soon come to its limit*; accordingly, we find that, when

when a body or system of bodies whose *weight* is (W), and the *distance of whose center of gyration from the axis* is (D), is put in motion by a *given weight* (P), the *greatest angular velocity* will be produced in the system in a given time, when P acts at a *distance from the axis of rotation* $= D\sqrt{\frac{W}{P}}$ [a]; and if $P = W$, i.e. if the system be put in motion by a weight equal to *its own weight*, then "the distance at "which P must act to produce the greatest angular "velocity, will be equal to *the distance of the* "*center of gyration* (D) from the axis."

LX.

([a]) By Art. 4 and 6, pp. 63, 64, when the system is put in motion by the *weight* (P), then $F = \frac{Px^2}{Px^2 + WD^2}$, $\therefore \frac{mFt}{\pi x} = \frac{mtPx}{\pi . Px^2 + WD^2}$, which varies as $\frac{x}{Px^2 + WD^2}$ when t is given; hence $\frac{x}{Px^2 + WD^2}$ = a *maximum*, $\therefore \dot{x} . \overline{Px^2 + WD^2} - 2Px\dot{x} = 0$, from which we have $x = D\sqrt{\frac{W}{P}}$. Also $Px^2 = WD^2$, $\therefore F = \frac{WD^2}{2WD^2} = \frac{1}{2}$.

In applying this to Exam. 1, page 70, we have $D = 20\sqrt{\frac{1}{2}}$, $W = 200$, $P = 15$, $\therefore x = 20\sqrt{\frac{200}{45}} = 42.6$; the *radius of the wheel A* therefore, upon which the given weight (W) of 15 *lbs* must act to communicate the *greatest angular velocity* to the paraboloid in a given time, is 42.6 *inches* = 3.51 *feet*; in which case (since $F = \frac{1}{2}$) the *number of revolutions performed in* $1'' = \frac{mFt}{\pi x} = \frac{16\frac{1}{12} \times 10}{3.14 \text{ \&c.} \times 3.51 \times 2} = 7.29$; consequently the number of revolutions in a *minute* = 60×7.29 = about 437

IX.

On the Maximum Effects produced by the action of one body upon another, by means of a string going over a fixed pulley.

In Example 4, page 41, we shewed the method of adjusting the length to the height of an inclined plane, so that a given weight descending *vertically* shall draw another given weight up it in *the least time possible*; and in Exam. 7, page 43, we determined the same when *both the bodies* moved upon inclined planes. We are now to ascertain the relation between the two bodies, so that the descending body shall produce the *greatest possible momentum in a given time* in the ascending one, whether the bodies move in a *vertical* direction or upon *inclined planes*.

1. By Art. 2, page 31, when a given body (P) draws up another body (x) by means of a string which goes over a fixed pulley, then the force which accelerates (P) in its descent (and consequently the force which accelerates (x) in its *ascent*) is equal to $\frac{P-x}{P+x}(F)$; the *velocity* of (x) generated in a given time will therefore be measured by $\frac{P-x}{P+x}$; hence the *momentum of* (x) (which varies as $Q \times V$) will be measured by $\frac{P-x \times x}{P+x}$. When this *momentum* is *greatest*, we have $\frac{Px-x^2}{P+x}$ = a *maximum*, from which

Px

$\dot{P}\dot{x}-2x\dot{x}\times\overline{P+x}-\dot{x}.\overline{Px-x^2}=0$; $\therefore x^2+2Px-P^2=0$, or $x=P.\sqrt{2}-1$; i. e. $P:x::1:\sqrt{2}-1::12:5$ very nearly [a].

2. By Art. 3, page 39, if a given weight (P), descending down an inclined plane whose *length* is (L) and *height* (H), draws another weight (x) up

[a] In this manner we ascertain the *magnitude of the load* in which a given weight, acting by means of a rope going over a fixt pulley, would generate the *greatest possible momentum* in a given time. But if it be required to find the *greatest load* which can be raised by a *given weight* through a *given space* (S) in a *given time*, then the portion to be raised at *each time* must be calculated in a different manner. Thus, let T=the *whole* time, and t=the time of *one* ascent, x=the portion to be raised at *each* ascent, and n=the *number of ascents*; then $T=nt$, and $n=\frac{T}{t}$. But $t=\sqrt{\frac{S}{mF}}$, $\therefore n=T\times\sqrt{\frac{mF}{S}}=\left(\text{for } F=\frac{P-x}{P+x}\right)$ $T\sqrt{\frac{m}{S}}\times\sqrt{\frac{P-x}{P+x}}$; now the *whole load* is equal to nx; hence $T\sqrt{\frac{m}{S}}\times\sqrt{\frac{P-x}{P+x}}\times x=$ a *maximum*; $\therefore\frac{Px^2-x^3}{P+x}=$ a *maximum*, or $\overline{2Px\dot{x}-3x^2\dot{x}}.\overline{P+x}-\dot{x}.\overline{Px^2-x^3}=0$; from which $x^2+Px-P^2=0$, and $x=\frac{P.\sqrt{5}-1}{2}$, or $P:x::2:\sqrt{5}-1::8:5$ very nearly. The *greatest quantity* of materials will therefore be raised by a given weight thus acting for a given time over a fixed pulley, "when the quantity raised at each ascent "is equal to about $\frac{5}{8}$*ths of that given weight*."

up an inclined plane of the *same* height whose *length* is (L'), then *the force which accelerates* (x) $= \frac{\overline{PL'-Lx}.H}{LL'.\overline{P+x}}$, which (in a given system of planes) *varies* as $\frac{PL'-Lx}{P+x}$; hence the *momentum of* (x) varies as $\frac{PL'x-Lx^2}{P+x}$; and if this be a *maximum*, then $\overline{PL'\dot{x}-2Lx\dot{x}} \times \overline{P+x} - \dot{x}.\overline{PL'x-Lx^2} = 0$; from which $x^2+2Px-\frac{P^2L'}{L}=0$, and $x = P.\sqrt{\frac{L+L'}{L'}-1}$, or $P : x :: 1 : \sqrt{\frac{L+L'}{L}}-1$[a].

3. If $L'=L$, then $P : x :: 1 : \sqrt{2}-1$; hence "if the planes be *similarly inclined to the horizon*, "then

(a) This, then, is the relation between P and x, when P generates the *greatest possible momentum* in (x) in a given time. If it were required to find what would be the *greatest quantity* of materials that could be raised in a given time by a given weight acting over a fixed pulley, placed at the top of two inclined planes of the same height and given lengths, then the *quantity to be raised at each ascent* may be calculated precisely in the same manner as in Note (*a*), page 95, by substituting "$\frac{PL'-Lx}{P+x}$," instead of "$\frac{P-x}{P+x}$" for F. To simplify the operation, let $L'=nL$, then $F=\frac{\overline{Pn-x}.L}{P+x}$, $\therefore \sqrt{\frac{Pn-x}{P+x}} \times x$ or $\frac{Pnx^2-x^3}{P+x}=$ a *maximum*; which gives $\overline{2Pnx\dot{x}-3x^2\dot{x}}.\overline{P+x}-\dot{x}.\overline{Pnx^2-x^3}=0$; from

" then the relation between the given weight (P) " and the weight to be raised, so as to produce the " *greatest possible momentum* in this latter weight " in a given time, is the same as when the bodies " act upon each other in a *vertical* direction;" See page 95. If P descends *vertically*, then $L=H$, $\therefore P : x :: 1 : \sqrt{\frac{H+L'}{L'}} - 1$.

LXI.

On the Maximum Effects produced by a given force acting upon a given weight by means of a wheel and axle, or a combination of wheels and axles.

1. Let the given *weight* P draw up another given weight (W) over a *wheel and axle*; in which the radius of the *wheel* : the radius of the *axle* :: $x : 1$; let the *weight* of the wheel and axle $= w$, and the *distance of their center of gyration* from the axis of rotation $= d$; then, by Art. 1, page 82, the *force which accelerates the ascent of* $W =$ (for $r=x$, $\rho=1$)

$$\frac{Px}{}$$

from which $x^2+\frac{3P-Pn}{2}.x-Pn^2=0$, and $x=\frac{P.\sqrt{9+10n+n^2}-3+n}{4}$; that " a given weight (descending down a plane whose *length* " is (L) and *height* (H)) may therefore draw the *greatest* " *quantity of materials* up another inclined plane of the *same* " height whose *length* is (nL) in a given time, the *weight raised* " *at each ascent* must bear to the *given weight* the ratio of " $\sqrt{9+10n+n^2}-3+n : 4$." If $n=1$ (or $L'=L$) this ratio becomes that of $\sqrt{5}-1 : 2$.

$\frac{Px-W}{Px^2+W+wd^2}$ [a] (F). Now, since $S=mFT^2$ and $T^2=\frac{S}{mF}$, the space described in a *given time* will be *greatest*, and the time of describing a *given space* will be *least*, when F is greatest, i.e. when $\dot{F}=0$; hence the *weight* W will be drawn up by the *weight* P through *the greatest space in a given time*, or through *a given space in the least time*, when the value of x is such that $\dot{F}=0$ [b], or when $x=\frac{W+\sqrt{W^2+PW+Pwd^2}}{P}$ [c].

2. If

(a) If the *inertia of* $P=0$, i.e. if the wheel be put in motion by the action of a *power* P, then $F=\frac{Px-W}{W+wd^2}$, which increases as x increases; in this case, therefore, the effect of P admits of no *maximum*, but keeps continually increasing as the distance from the axis increases.

(b) Since $F=\frac{Px-W}{Px^2+W+wd^2}$; when F is a *maximum*,
$\dot{F}=P\dot{x}.\overline{Px^2+W+wd^2}-2Px\dot{x}.\overline{Px-W}.=0$,
$\therefore Px^2-2Wx-\overline{W+wd^2}=0$; which equation, solved, gives $x=\frac{W+\sqrt{W^2+PW+Pwd^2}}{P}$.

In applying this to Exam. 1, page 87, we have $P=100$, $W=500$, $w=80$, $d=3$ feet; $\therefore x=\frac{500+\sqrt{372000}}{100}=11.09$; hence the *radius of the wheel*, by which a weight of 100 *lbs* would raise a weight of 500 *lbs* (under all the circumstances mentioned in this example) through a given space in *the least time possible*, is about 11 feet.

But this mode of estimating the *maximum effect* produced by a wheel and axle (in which, neither the *materials of which it is composed*,

2. If the *weight of the wheel axle* be not taken into the consideration, then $w=0$, $\therefore x=\frac{W}{}$

composed, nor the *shape and relative magnitude* of its different parts, are taken into the account) is extremely vague and indefinite, and, without certain *limitations*, will often lead to conclusions in their nature totally impracticable. A wheel and axle, in which *the diameter of the wheel is* 22 *feet*, used for the ordinary purpose of *raising weights*, would be at best a very *inconvenient* machine; but, besides this, the *strain* and *stress* upon such a wheel must be so great as to make it continually liable to *fracture*. Before this theory therefore can be applied to practice, the weight and quality of the materials of which the machine is composed must be brought under consideration, and the different parts of it so arranged and adjusted as to give it *strength* and *stability*.

Suppose, then, it was required "to construct, out of *a mass of* "*iron weighing* 80 *lbs*, a wheel and axle which, both as to the "*strength* and *power* of the machine, shall be perfectly adapted "to the purpose of raising *a weight of* 500 *lbs*, by means of *another* "*weight of* 100 *lbs*, through a given space, in the least time "possible."

It has already been mentioned, in the 11th Lecture, that a bar of iron *one foot long and an inch square* (*weighing about* 5 *lbs*) is just able to sustain a weight of 2190 *lbs*; and proceeding upon the principles laid down in that Lecture, it will be found that a *cylindrical* bar *one foot long and diameter* 2 *inches* will sustain full *six times* that weight. We may therefore reasonably conclude, that if a part of the given mass, amounting to about 15 *lbs* (which is the weight of the *cylindrical bar* just alluded to), be appropriated to forming the *axle*, this axle will be capable of sustaining a weight of 680 *lbs*, without the slightest tendency to *bending* or *fracture*. If the remaining 65 *lbs* be made into a cylindrical wheel *one inch thick*, the *radius* of that wheel will be about 7 *inches*, and the distance of the center of gyration of *it and the axle* from the axis of rotation will be 4.4 *inches*; suppose the center of gyration

$\frac{W+\sqrt{W^2+PW}}{P}$; and if $W=nP$, then $x=n+\sqrt{n^2+n}$. If $n=1$, then $x=1+\sqrt{2}$; hence if the weight applied to the *wheel* be equal to the weight suspended on the *axle*, then the latter will be made to ascend, by the action of the former, through a given space in the least time possible, when the radius of the *wheel* : the radius of the *axle* :: $1+\sqrt{2}$: 1 (:: 12 : 5 nearly).

3. If it were required to find the magnitude of a weight in which a given weight (P), acting upon a wheel

gyration to be brought within 3 *inches* of the axis, whilst the *weight* of the wheel remains the same, then this *cylindrical* wheel may be converted into another of a *greater radius*, and of a form somewhat *tapering* from the center to the circumference, without any great diminution of its *strength*. To find the diameter of this wheel when it will produce its *maximum effect*, we have $W=500$, $P=100$, $w=80$, $d=3$ inches, $\therefore x=11.09$ inches; and consequently the *diameter* = about 22 inches. In this case $F=\frac{Px-W}{Px^2+W+wd^2}=\frac{1}{22.1}$; the *shortest time*, therefore, in which P will draw up W through a given space (S) is $\left(\sqrt{\frac{S}{mF}}\text{ or}\right)$ $\sqrt{\frac{22.1S}{m}}$ seconds.

(*) The value of x thus found will evidently apply to the case of a straight lever (without weight) *just as it begins to move*; i. e. let the *longer* arm of a straight lever : the *shorter* :: x : 1, then a power (P), acting upon the longer arm, will make a weight (W), attached to the shorter arm, move through a small space in *the least time possible* when $x=\frac{W+\sqrt{W^2+PW}}{P}$, and *vice versâ*.

wheel and axle (whereof the radius of the *wheel* : radius of the *axle* :: r : 1), should generate the *greatest possible momentum* in a given time; then let that weight $= x$, and the expression for the *accelerative force with which it ascends* is $\frac{Pr - x}{Pr^2 + x + wd^2}(F)$, w and d representing, as before, the *weight* and *distance of the center of gyration* of the given wheel and axle. Now $V = 2mFt$, $\therefore$ the *velocity generated in a given time* varies as F, and consequently the *momentum* generated in a given time varies as Fx, or as $\frac{Prx - x^2}{Pr^2 + x + wd^2}$, which is a *maximum* when $x = \sqrt{P^2r^3.r+1 + wd^2.Pr + 2Pr^2 + wd^2} - \overline{Pr^2 + wd^2}$.[b]

4. If the weight of the wheel and axle be not taken into the consideration, then $w = 0$, and $x = Pr\sqrt{r.r+1} - Pr^2$[c]; and if the radius of the *axle* be

(b) For $\dot{F} = \dot{x}.\overline{Pr - 2x}.\overline{Pr^2 + x + wd^2} - \dot{x}.\overline{Prx - x^2} = 0$, from which $x^2 + 2.\overline{Pr^2 + wd^2}.x - \overline{P^2r^3 + Prwd^2} = 0$, which equation solved gives $x = \sqrt{P^2r^3.r+1 + wd^2.Pr + 2Pr^2 + wd^2} - \overline{Pr^2 + wd^2}$.

Take, for example, an *iron* wheel and axle, in which $w = 10\,lbs$, $r = 3$ inches, $d = 1.2$ inches; and let $P = 10\,lbs$; then $x = \sqrt{14031.36} - 104.4 = 14\,lbs$ very nearly; the *greatest momentum* therefore which could be generated in a given time by a weight of $10\,lbs$, acting upon this wheel and axle, would be, when the *weight raised* was about 14 pounds.

(c) The value of x here found, as well as that in Art. 2, will evidently apply to the case of a straight lever, *just as it begins to move;*

be equal to the radius of the *wheel*, then $r=1$, and $x=P.\sqrt{2}-1=.414\,P=$ about $\frac{5}{12}$ths P; hence, when the *radius of the axle* is equal to the *radius of the wheel*, the *greatest possible momentum* will be generated by a given weight in a given time, when the *weight raised by the axle* is about $\frac{5}{12}$ths of the *weight which acts upon the wheel*; which indeed it ought to be, as, in this case, the machine becomes analogous to the *fixed pulley* (see Art. 1, page 94)[n].

5. In

move; i. e. let *one* arm : the *other* :: r : 1; then if a power (P) be applied at the extremity of the arm whose length is (r), and a weight (x) be attached to the extremity of the other arm, the *greatest possible momentum* will be generated in this weight (in ipso motûs initio) when $x=Pr.\sqrt{r.r+1}-r$.

(n) By substituting $\frac{Pr-x}{Pr^2+x}$ (which is the value of F, when $wd^2=0$) for F in the value of nx, Note (*a*), page 95, we may also determine the quantity of materials to be raised at each ascent by the *wheel and axle*, when the whole quantity to be raised by the given weight (P) in a given time is a *maximum*. In this case $\sqrt{\frac{Pr-x}{Pr^2+x}}\times x=$a *maximum*; which quantity is very similar to the quantity $\sqrt{\frac{Pn-x}{P+x}}\times x$ in Note (*a*), page 96, and by a similar process x is found to be equal to $\frac{P.\sqrt{9r^4+10r^3+r^2}-3r^2+r}{4}$; hence, *the weight raised at each ascent* : P :: $\sqrt{9r^4+10r^3+r^2}-3r^2+r$: 4.

5. In order to estimate the maximum effect of a *combination of* (n) *wheels and axles*, in which the radius of *each wheel* : the radius of *each axle* :: $x : 1$ [b], and which is put in motion by a given *power* (P), we must have recourse to Note (c), page 87, where *the force which accelerates the ascent of* W will (in this case) be represented by $\frac{Px^n - W}{W + wd^2 + wd^2x^2 + \&c.... wd^2x^{2n-2}}$ (F); the nature of this expression will be best understood, by substituting for (n) the numbers 1, 2, 3, 4, &c. in succession.

6. If $n = 1$, then $F = \frac{Px - W}{W + wd^2}$, which coincides with the value ascribed to F in Note (a), page 98; and, as was there observed, it admits of no *maximum*, since the effect of P keeps continually increasing as the distance at which it acts from the axis increases.

7. If $n = 2$, then $F = \frac{Px^2 - W}{W + wd^2 + wd^2x^2}$. This also is an expression which admits of no *maximum*, as it keeps continually increasing as x increases, but *never exceeds a certain limit*; for let x be *infinite*, then

(b) In this and the remaining articles to the end of this section we include, under the denomination of the *wheel and axle*, not only "the *wheel D* and *axle E*," but "the *large wheel B* and "*small wheel C*;" "the *handle PQ* and *small wheel A*;" &c. &c. (See Figure in page 86.)

then $F=\frac{Px^n}{wd^n x^n}=\frac{P}{wd^n}$. Hence, if a machine consists of two wheels and axles of this kind, the contrivance from of W keeps continually increasing as P is applied at a greater distance for the axis, but the limit to which it approaches (and which it can never exceed) is that part of the force of gravity which is expressed by the quantity $\frac{P}{wd^n}$ [a].

5. If n be equal to or greater than 3, then the index of x in the denominator of the fraction (F) is greater than the index of x in the numerator, and the fraction itself admits of a maximum; hence, if the machine consists of three or more wheels, it will arrive at its maximum effect when the common radius of the wheels is of some determined magnitude, which may be found by making $\dot{F}=0$. But to find the value of x in this case, will require the solution of an equation of at least $2n-1$ dimensions [b].

9. If

(a) Let the machine, for instance, consist of two such wheels and axles as are described at the end of Note (b), p. 100; then $P=100, w=60, d=3, \therefore F=\frac{P}{wd^2}=\frac{100}{720}=\frac{5}{36}$; and the utmost force which could be exerted by (P) applied to such a machine, would never be sufficient to draw up (W) through a space of 20 feet in $\left(\sqrt{\frac{8}{mP}}\right)\sqrt{\frac{20\times 36}{10\frac{1}{2}\times 5}}$, or 2.9 seconds.

(b) By way of example, suppose the machine to consist of *three* wheels, and that $W=500$ *lbs*, $P=35$ *lbs*, $w=10$, $d=2$ inches; then

9. If the radii of these (n) wheels and axles be not to each other in the *same* ratio, but if (beginning with that part of the machine to which W is appended) they are severally expressed by the ratios of $x : 1$, $y : 1$, $z : 1$, $v : 1$, &c; and if the corresponding values of their *weights* and *distances of their centers of gyration from the axes* be represented by w, w', w'', w''', &c.; d, d', d'', d''', &c.; then, by referring to the process of Art. 3, page 86, and its acompanying *Note*, it might easily be shewn, that the *force which accelerates the ascent of* W $(F) = \frac{Pxyzv \text{ &c.} - W}{W + wd^2 + w'd'^2x^2 + w''d''^2x^2y^2 + w'''d'''^2x^2y^2z^2 + \text{&c.}}$, the law of which expression is manifest; and if the circumstances under which the machine is constructed be such as to afford $n-1$ equations (involving some or all of the quantities x, y, z, &c.) besides that of $\dot{F}=0$, then the values of x, y, z, v, &c. may be determined[b].

then $F=\frac{Px^3-W}{W+wd^2+wd^2x^2+wd^2x^4}=\frac{7x^3-100}{108+8x^2+8x^4}$; and the equation arising from making $\dot{F}=0$ is, $x^5-x^3-57.1x^2-40.5x-28.5=0$, in which the value of x is found, by one approximation, to be 4.66 inches.

(b) Suppose, for instance, that the machine consists of *two* wheels, and that it be required to find its *maximum effect*, under the condition that the *sum of the radii* of these wheels shall be equal to the given quantity (a). In this case $F=\frac{Pxy-W}{W+wd^2+w'd'^2x^2}$; and $\dot{F}=0$, $x+y=a$, are two equations by which the values of x and y may be determined. Or, if there be *three* wheels, and *another* condition be added, such as, for

instance, "that the radius of the *third* wheel shall exceed "twice the radius of the *first* wheel by the given quantity (b);" then $F=\frac{Pxyz-W}{W+m^2-n^2c^2x^2+n^2c^2x^2y^2}$, and $\dot{F}=0$, $x+y+z=a$, $z=2x+b$, are three equations for finding the value of x, y, z. In like manner might this Theorem be applied to finding the dimensions of a machine consisting of *any number* (n) of wheels and axles; provided that, in every case, the relative values of m, m', m'' &c.; d, d', d'' &c.; be so adjusted as to accord with the equations which may be formed for the purpose of expressing certain relations between the *radii* (x, y, z, &c.) *of the wheels*. To effect this for *each component part* of the machine, we must have recourse to the principles laid down in Note (k), page 99.

Lecture XVII.

ON

THE METHOD OF FINDING THE CENTERS OF OSCILLATION, AND PERCUSSION, OF A BODY OR SYSTEM OF BODIES.

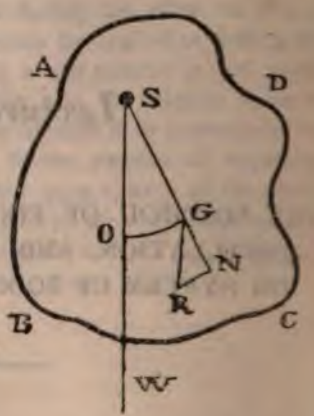
ALTHOUGH the centers of *oscillation* and *percussion* of a body, or system of bodies, are found to exist in the *same point* of the body or system; yet, as their *properties* are not the same, they will each require a distinct process of investigation. We shall begin with the center of *oscillation*.

LXII.

On the method of investigating the general expression for the distance of the Center of Oscillation, *of a body or system of bodies, from the axis of suspension.*

1. From what has been observed in Art. 8, p. 101, Part I, it appears, that if a body or system of bodies (repre-

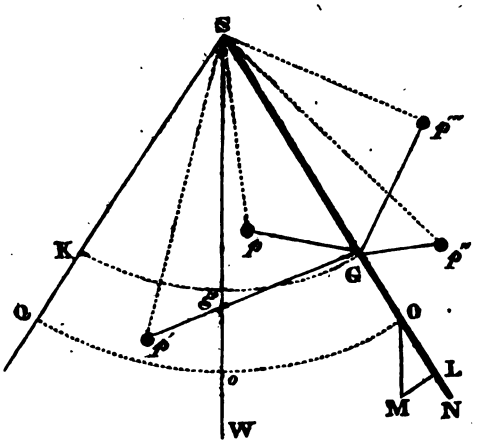
(represented by the Figure *ABCD*) whose *center of gravity* is *G*, and which is supported by an *horizontal axis of suspension* passing through *S*, be brought into the position *SGN*, it will always have a tendency to descend till the line *SGN* coincides with the *vertical* line *SOW*. From the same article it also appears, that if *GR* be drawn parallel to *SOW*, and *RN* perpendicular to *SN*, then the *force* which produces this tendency in any *given position SGN* will bear to the *weight of the body or system* the ratio of *RN* : *GR*; let this *force*=ϕ, the *weight of the body or system*=w, and the *angle of position OSG* or *RGN*=α, then $\phi : w :: RN : GR ::$ sin. α : rad. (1), $\therefore \phi = w \times$ sin. α.



2. In a *given* body or system, $\phi \propto$ sin. α, and consequently *decreases* whilst the line *SG* moves from its *oblique* position *SGN* to the *vertical* position *SOW*. But since this force acts *continually*, the *velocity* generated in the body or system will continually *increase* till *SG* arrives at the position *SOW*, when it will be *greatest;* and if there were no *friction upon the axis of suspension*, nor *resistance from the air*, it is evident that the velocity thus acquired would be just sufficient to make the point *G* describe an arc equal to *GO* on *the other side* of *SOW*,

SOW, with a motion continually retarded. When the center of gravity has arrived at the *upper extremity* of this latter arc, it will begin to descend again; and so the body or system would go on vibrating *ad infinitum*, its center of gravity describing, during each vibration, *an arc equal to* $2GO$ round S as a center.

3. Suppose now any number of small particles of matter, p, p', p'', p''', &c. (connected with each other and with their *common center of gravity* (G) by inflexible rods void of gravity), to be suspended



from an horizontal axis passing through the point S by an inflexible rod (SG) void of gravity; and let them vibrate in such a manner that SG may be always in *the same vertical plane*. Produce SG to N; and describe the circular arc Gg meeting the vertical line SW (drawn in the plane passing through SN) in

in g. From p, p', p'', &c. draw the lines Sp, Sp', Sp'', &c. in planes respectively parallel to the plane passing through SG, and meeting the axis of suspension (in different points) S. Let $p + p' + p'' + \&c. = w$, and $\angle WSN = \alpha$; then, (by Art. 1,) when the system is in the position SGN, the *moving force* acting upon the point $G = w \times \sin. \alpha$; and supposing the *inertia* of the particles p, p', p'', &c. to be estimated as if they were all collected in G, then (Note a, page 51) the *mass moved* will be $\frac{p \times Sp^2 + p' \times Sp'^2 + p'' \times Sp''^2 \&c.}{SG^2}$; hence the *accelerative force* (F) with which the point G begins to vibrate, is $\frac{w \times \sin. \alpha \times SG^2}{p \times Sp^2 + p' \times Sp'^2 + p'' \times Sp''^2 + \&c.}$; and by Note a, page 68, its *angular velocity* will be measured by $\left(\frac{F}{r}\right) \frac{w \times \sin. \alpha \times SG}{p \times Sp^2 + p' \times Sp'^2 + p'' \times Sp''^2 + \&c.}$

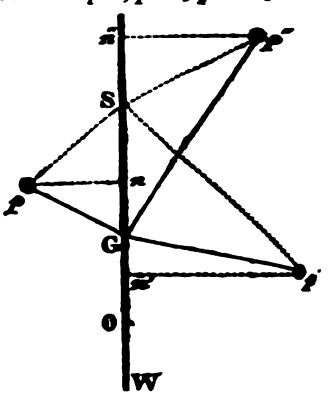
4. Conceive now all the matter of the system (w) to be concentrated in any point O of the line SN; draw OM parallel to SW, and ML at right angles to SN; then, for the same reason as in Art. 1, the *moving force upon* the point O : w :: ML : MO :: sin. $\angle MOL$ or WSN (sin. α) : rad. (1), $\therefore$ the *moving force* upon the point O in this case is equal to $w \times sin. \alpha$; and as the *mass moved* is w, the *accelerative force* $= \frac{w \times \sin. \alpha}{w} = \sin. \alpha$; consequently the *angular velocity* of the point $O = \left(\frac{F}{r}\right) \frac{\sin. \alpha}{SO}$.

5. Let

5. Let $\frac{w \times \text{sin. } \alpha \times SG}{p \times Sp^2 + p' \times Sp'^2 + p'' \times Sp''^2} = \frac{\text{sin. } \alpha}{SO}$, then $SO = \frac{p \times Sp^2 + p' \times Sp'^2 + p'' \times Sp''^2 + \&c.}{w \times SG}$; in which case, O is such a point, that, if all the matter of the system were collected in it, the system would *begin* to vibrate with the *same angular velocity* as if it were spread through the system. But, since the relation of the *angular velocities* of the points G and O, under the foregoing circumstances, will be the same, *whatever be the position of the line* SGN, it is evident that these points will not only *begin*, but also *continue*, to move in such a manner as to perform their *semi-vibrations* through the angle oSO, and their *entire vibrations* through the angle QSO $(= 2oSO)$, with the same *angular velocity*, and consequently in the same *time*. This point O is called the *center of oscillation* of the system of particles p, p', p'' &c.; and may be defined to be "that point, into which if all the matter "of the system were collected, and acted upon by "the same force which impels the center of gravity "of the system, the matter thus concentrated "would perform its vibrations in the *same time* as "if each particle retained its original position in the "system."

6. Suppose the situation of the particles p, p', p'' &c. to be referred to the vertical plane passing through SG,

SG, and let fall the perpendiculars $pn, p'n', p''n''$, &c. upon the line SGW; then, by Euc. II, 12 and 13, we have,



$$p \times Sp^2 = p \cdot \overline{SG^2 + pG^2 - 2SG \times Gn}$$
$$p' \times Sp'^2 = p' \cdot \overline{SG^2 + p'G^2 + 2SG \times Gn'}$$
$$p'' \times Sp''^2 = p'' \cdot \overline{SG^2 + p''G^2 - 2SG \times Gn''}$$
$$\&c. = \&c.$$

Therefore $p \times Sp^2 + p' \times Sp'^2 + p'' \times Sp''^2$ &c. $=$
$\overline{p + p' + p''}\ (w) \cdot SG^2 + p \times pG^2 + p' \times p'G^2 + p'' \times p''G^2$ &c. (a);

Hence, $\dfrac{p \times Sp^2 + p' \times Sp'^2 + p'' \times Sp''^2 \&c.}{w \times SG} = (SO = SG + GO =)$

$$\frac{w \times SG^2 + p \times pG^2 + p' \times p'G^2 + p'' \times p''G^2 \&c.}{w \times SG} =$$

$$SG + \frac{p \times pG^2 + p' \times p'G^2 + p'' \times p''G^2 \&c.}{w \times SG};$$

consequently $GO = \dfrac{p \times pG^2 + p' \times p'G^2 + p'' \times p''G^2 \&c.}{w \times SG}$.

7. From hence it appears, that $SG \times GO = \dfrac{p \times pG^2 + p' \times pG'^2 + p'' \times p''G^2}{w} =$ a *constant quantity* (a^2); $\therefore GO$ varies *inversely* as SG; i.e. "in a "given system of particles, p, p', p'', &c. the distance "of

(a) For by the property of the *center of gravity* (supposing a plane to pass through G at right angles to SGW) $p' \times Gn' = p \times Gn + p'' \times Gn''$, $\therefore p' \times 2SG \times Gn' = p \times 2SG \times Gn + p'' \times 2SG \times Gn''$, or $-p \times 2SG \times Gn + p' \times 2SG \times Gn' - p'' \times 2SG \times Gn'' = 0$.

" of the center of *oscillation* from the center of *gra-*
" *vity* varies inversely as the distance of the *point*
" *of suspension* from the same;" if, therefore, the point of suspension (S) be removed to a considerable distance from the center of gravity (G), the interval GO will become *very small*; and if it be removed to an infinite distance, then the centers of oscillation and gravity will actually *coincide*.

8. Since $SG \times GO = a^2$, we have $SG = \frac{a^2}{GO}$; $\therefore$ $SO = SG + GO = \frac{a^2}{GO} + GO = \frac{a^2 + GO^2}{GO}$. Suppose now the center of *oscillation* to be made the point of *suspension*, and that d = the distance of the center of *oscillation* from the center of *gravity*, then, by Art. 6, $d = \frac{a^2}{OG}$, $\therefore$ $OG + d = OG + \frac{a^2}{OG} = \frac{a^2 + OG^2}{OG}$ $= SO$; when the point of suspension therefore is in O, the center of oscillation will be in S. Hence, if the plane of vibration[b] remains the same, the system p, p', p'', &c. will perform its vibrations in the same time and manner, when its position (as referred to that plane) is *inverted*, and the *center of oscillation* is become the *point of suspension*.

9. Moreover, since $SG \times GO$ = a *constant quantity*, if

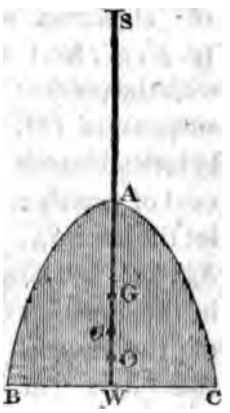
(b) By the *plane of vibration* is meant " the plane passing " through the line which joins the point of suspension and the " center of gravity, being that in which the *center of gravity* " always vibrates."

if SG be *given*, then GO (and consequently SO) must be given *also*; i. e. "so long as the distance "between the *point of suspension* and *center of gravity* "of a given system of particles p, p', p'', &c. remains "the same, *the distance of the center of oscillation* "*from the point of suspension* will remain the same;" the number of ways therefore in which a given body or system of bodies may be suspended, so that its center of oscillation shall always be at the same distance from the point of suspension, is *unlimited*; for if lines be made to branch out from the center of gravity like the *radii of a sphere*, and the body or system be suspended from a point (at a given distance from the center of gravity) in the direction of any one of these lines, the center of oscillation will in each case be at the same distance from the point of suspension.

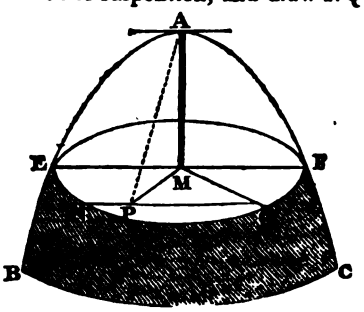
10. If the number of the particles p, p', p'', &c. be increased ad infinitum, so that the system becomes a *solid body*, then its center of oscillation may be ascertained, by finding the value of $\frac{p \times Sp^2 + p' \times Sp'^2 + p'' \times Sp''^2, \text{ \&c.}}{w \times SG}$ from the nature and properties of the given body. Let the body be a solid formed by the revolution of an algebraic curve about its axis; let the point of suspension be in the *vertex*, and let the plane of vibration pass through the *axis*, of the solid; then, if $x =$ the *abscissa*, $y =$ the corresponding *ordinate*, $\pi = 3.141$, &c. $s =$ the *content of the solid*, $g =$ the distance of its *center of gravity* from the vertex,

$\frac{f^c \pi y^4 x^2 \dot{x} + \frac{1}{4}\pi y^4 \dot{y}}{sg} (= AO)$ will be *the distance of the center of oscillation from the vertex of the solid*[a].

11. If the point of suspension be not in the *vertex* (A) of the solid, but in the *axis produced* (S); suppose ABC to represent a section of the solid cut through its axis AW; let G be its *center of gravity*, O the center of oscillation *when the body is suspended from* A, o the center of oscillation *when it is suspended from* S; then, by Art. 7th, $AG \times GO = SG \times Go$; $\therefore$ $Go = \frac{AG \times GO}{SG}$, and $So = SG + Go = SG + \frac{AG \times GO}{SG}$ *equals*

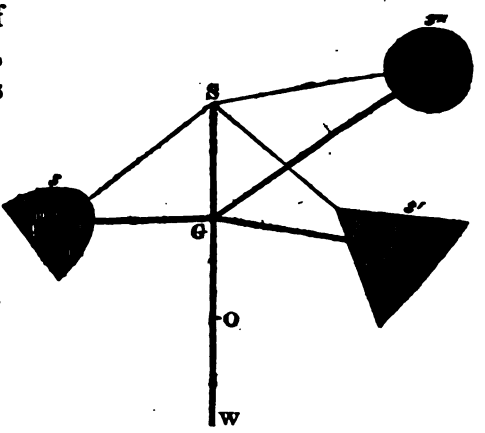


(a) Let BAC represent the solid as suspended from its *vertex* A; and let $ENQF$ be a *circular* section of it perpendicular to the *axis* AM; suppose EF to be that diameter of the circle $ENQF$, which is parallel to the axis of suspension, and draw NQ *parallel*, MP *perpendicular* to it; join AP, MQ. For the purpose of estimating the value of $p \times Sp^2 + p' \times Sp'^2 + p'' \times Sp''^2$, &c. ($M$) for the circular plane $ENQF$, let $AM = a$, $MQ = r$, $MP = v$; then $AP^2 = a^2 + v^2$, $NQ = 2\sqrt{r^2 - v^2}$, and the fluxion of the area $ENPQF$ ($\dot{A}$) $= 2\dot{v}\sqrt{r^2 - v^2}$; hence $\dot{M} = 2\dot{v}\sqrt{r^2 - v^2} \times \sqrt{a^2 + v^2} = \frac{2a^2}{}$



= the *distance of the center of oscillation from the point of suspension S.*

12. Suppose now s, s', s'', &c. to be any number of algebraic solids, whose centers of gravity (g, g', g'', &c.) are connected with each other, and with the point of suspension (S), by inflexible rods void of gravity; let the lines Sg, Sg', Sg'', &c. be in the direction of the respective *axes* of these solids, and let the Figure $S\,g\,g'\,g''$, as well as the line SGW (passing through their *common center*



$2a^2\dot{v}\sqrt{r^2-v^2}+2v^2\dot{v}\sqrt{r^2-v^2}$, $\therefore M=a^2A+f'2v^2\dot{v}\sqrt{r^2-v^2}=a^2A+\frac{1}{4}r^2A-\frac{1}{2}v.\overline{r^2-v^2}\big|^{\frac{3}{2}}$* = (when $v=r$) $\overline{a^2+\frac{1}{4}r^2}\times$ the *semi-circle* $ENQF$ = (for the *whole circle*) $\overline{a^2+\frac{1}{4}r^2}\times\pi r^2=\pi a^2r^2+\frac{1}{4}\pi r^4$. Suppose now that $AM=x$, EM or $MQ=y$, then the quantity (M) for the *circular section* $ENQF$ would be equal to $\pi x^2y^2+\frac{1}{4}\pi y^4$; and consequently for the *solid* it would be equal to $f'\overline{\pi y^2x^2\dot{x}+\frac{1}{4}\pi y^4\dot{x}}$, when x is made equal to *the height of the solid.*

* Assume $v.\overline{r^2-v^2}\big|^{\frac{1}{2}}=P$, then $\dot{v}.\overline{r^2-v^2}\big|^{\frac{1}{2}}-3v^2\dot{v}\sqrt{r^2-v^2}=r^2\dot{v}\sqrt{r^2-v^2}-4v^2\dot{v}\sqrt{r^2-v^2}=\frac{1}{2}r^2\dot{A}-4v^2\dot{v}\sqrt{r^2-v^2}=\dot{P}$; hence $2v^2\dot{v}\sqrt{r^2-v^2}=\frac{1}{4}r^2\dot{A}-\frac{1}{4}\dot{P}$, $\therefore f'2v^2\dot{v}\sqrt{r^2-v^2}=\frac{1}{4}r^2A-\frac{1}{4}P$.

center of gravity G), lie in the *plane of vibration*. Let Sg, Sg', Sg'', &c. $= g, g', g''$, &c.; and having found the *centers of oscillation* o, o', o'', &c. by the rule laid down in the preceding article, let So, So', So'', &c. $= d, d', d''$, &c.; then, supposing any one of these solids (s for instance) to be composed of particles p, p', p'', &c. whose distances from S are Sp, Sp', Sp'', &c. So (by Art. 5) will be equal to $\frac{p \times Sp^2 + p' \times Sp'^2 + p'' \times Sp''^2}{sg}$ &c. $= d$, $\therefore p \times Sp^2 + p' \times Sp'^2 + p'' \times Sp''^2$ &c. $= sdg$. For the same reason, "the sum of the products arising from multiplying each particle of the solids s', s'', &c. into "the square of its distance from S," will be $s'd'g'$, $s''d''g''$, &c.; the *sum of these products* therefore, calculated for *all* the solids, will be $sdg + s'd'g' + s''d''g''$ + &c. Hence, if O be *the center of oscillation of this system of bodies*, then will SO (by Art. 5) be equal to $\frac{sdg + s'd'g' + s''d''g'' + \text{&c.}}{s + s' + s'' + \text{&c.}\, SG}$ [a].

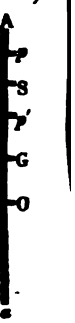
EXAM. I.

To find the center of oscillation (O) *of a right line or small cylinder* (Aa), *suspended from any point* (S) *above its center of gravity* (G).

Let

[a] If the bodies be small, and at a considerable distance from S, then (by Art. 7) g, g', g'', &c. are *very nearly* equal to d, d', d'', &c. In this case, therefore, SO may be assumed equal to $\frac{sg^2 + s'g'^2 + s''g''^2 + \text{&c.}}{s + s' + s'' + \text{&c.}\, SG}$ without any material error.

Let $Sp=x$, $Sp'=y$, then the *sum of each particle, multiplied into the square of its distance*, reckoned from S to A, is $\int x^2\dot{x}=\frac{1}{3}x^3=\frac{1}{3}SA^3$ (when $x=SA$); for the same reason, this sum *reckoned from S to a* will be equal to $\frac{1}{3}Sa^3$; now, by the property of the center of gravity[a], $Aa\times SG=Sa\times\frac{1}{2}Sa-SA\times\frac{1}{2}SA=\frac{1}{2}.\overline{Sa^2-SA^2}$; hence $SO=\frac{\frac{1}{3}Sa^3+\frac{1}{3}SA^3}{Aa\times SG}=\frac{2.\overline{Sa^3+SA^3}}{3.\overline{Sa^2-SA^2}}$.



Cor. 1. If $SA=o$, i.e. if the line be suspended from its *extremity*, then $SO=\frac{2Aa^3}{3Aa^2}=\frac{2}{3}Aa=\frac{2}{3}da$ *the length* of the line; and if $Sa=nSA$, then $SO=\frac{2.\overline{n^3+1}}{3.\overline{n^2-1}}SA$. Let $n=2$, then $SO=2SA$, and the center of oscillation is at the *lower extremity* of the line[b].

Cor. 2. Since $Sa=nSA$, $Aa=Sa+SA=\overline{n+1}.SA$; $\therefore$ the distance between the center of oscillation

(a) Cor. to Exam. 5, page 108, Vol. I.

(b) This follows indeed from Art. 8; for since, when the line is suspended from *one* extremity, the center of oscillation is at a distance from the *other* equal to $\frac{1}{3}$d the whole length of the line; if the *center of oscillation* be made the *point of suspension*, then the center of oscillation of the line thus *inverted* will be at its *lower extremity*, and SO will be equal to $2SA$.

oscillation and point of suspension, when the line is suspended *from its extremity A* : the distance between the same when it is suspended *from any point S* (so placed that $Sa=nSA$) :: $(\frac{2}{3}Aa)$ $\frac{2.\overline{n+1}}{3}SA : \frac{2.\overline{n^3+1}}{3.\overline{n^2-1}}SA :: \overline{n+1}.\overline{n^2-1} : n^3+1.$



Cor. 3. If a line SA of *uniform thickness*, but the *density* of whose particles increases as their *distance from S* increases, be suspended from its extremity (S), then $SO=\frac{f^t.x^3\dot{x}}{f^t.x^2\dot{x}}$ [(c)] $=\frac{\frac{1}{4}x^4}{\frac{1}{3}x^3}=$ (when $x=SA$) $\frac{3}{4}SA$.

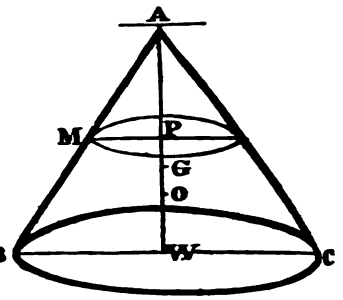
EXAM. II.

To find the center of oscillation of a cone (ABC) *suspended from its vertex* (A); *its axis* (AW) *lying in the plane of vibration.*

Let

(c) For let $Sp=x$, and let the density at some *given* distance (1) from $S=d$, then the density *at the distance x* will be dx; the *weight* of a particle, therefore, at the distance x will be $dx\dot{x}$, and consequently the *sum of each particle multiplied into the square of its distance from S*, calculated for the part Sp, will be $f^t.dx\dot{x}\times x^2$ or $f^t.dx^3\dot{x}$. Let g be the *center of gravity* of the part Sp, then (by Art. 1, page 129, Vol. I.) $Sg=\frac{f^t.dx^2\dot{x}}{\text{weight of } Sp}$; $\therefore$ *weight of* $Sp\times Sg=f^t.dx^2\dot{x}$; hence $SO=\left(\frac{f^t.dx^3\dot{x}}{\text{weight of } Sp\times Sg}=\right)\frac{f^t.dx^3\dot{x}}{f^t.dx^2\dot{x}}=\frac{f^t.x^3\dot{x}}{f^t.x^2\dot{x}}$.

Let $AP=x$, $PM=y$, the *height* of the cone $=h$, the *radius of its base* $=nh$, then $x : y :: h : nh$, $\therefore y=nx$; by Art. 10, $AO=$



$$\frac{\int \pi y^2 x^2 \dot{x} + \frac{1}{4}\pi y^4 \dot{x}}{sg} = \frac{\int \pi n^2 x^4 \dot{x} + \frac{1}{4}\pi n^4 x^4 \dot{x}}{sg} = \frac{\frac{1}{5}\pi n^2 x^5 + \frac{1}{20}\pi n^4 x^5}{sg}$$

= (for the *whole cone*; for $g=\frac{3}{4}h$, $s=\frac{1}{3}\pi n^2 h^3$) $\frac{\frac{1}{5}\pi n^2 h^5 + \frac{1}{20}\pi n^4 h^5}{\frac{1}{4}\pi n^2 h^4} = \frac{4+n^2}{5}.h.$

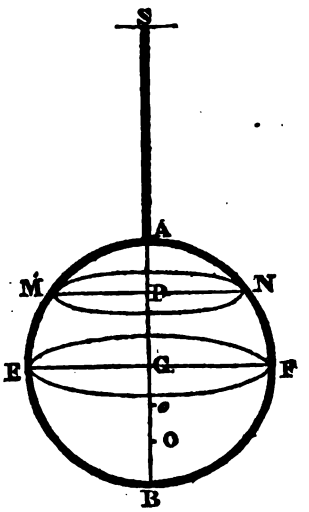
Cor. 1. If $n=1$, (in which case $AW=WB$), then $AO=h=AW$; hence, in a *right-angled* cone, the center of oscillation is in the *center of its base*. If n be *greater* than 1 (and $\therefore WB$ greater than AW), AO is greater than AW; in an *obtuse-angled* cone, therefore, the center of oscillation falls *below the base*. If n be *less* than 1 (or WB less than AW), then AO is *less* than AW; hence, in the *acute-angled* cone, and in that *only*, the center of oscillation falls *above the base*, i. e. *within the cone*.

Cor. 2. Since, in a *right-angled* cone, suspended from the vertex, the center of oscillation falls in the center of the base; it follows, from Art. 8, that if a cone of this kind be suspended from the *center of its base*, the center of oscillation will be in the *vertex*.

Ex.

EXAM. III.

To find the center of oscillation of a sphere (ABF) *suspended from a point* (S) *at any distance from its surface.*



Let $AP=x$, $PM=y$, $r=$ *radius* of the sphere; and let O be the center of oscillation when *it is suspended from* A, o when *it is suspended from* S; then AO

$$=\frac{f\dot{x}\sqrt{\pi y^2 x^2 \dot{x}+\frac{1}{4}\pi y^4 \dot{x}}}{sg}=$$

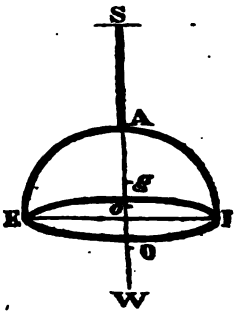
$$\frac{f\dot{x}\sqrt{\pi x^2 \dot{x}.2rx-x^2+\frac{1}{4}\pi \dot{x},2rx-x^2}^2}{sg}$$

$$=\frac{\frac{1}{2}\pi r^2 x^3+\frac{1}{4}\pi r x^4-\frac{2}{10}\pi x^5}{sg}=$$

(when $x=2r$, in which case $s=\frac{4}{3}\pi r^3$, $g=r$) $\frac{7r}{5}$; $\therefore GO=AO-AG=\frac{2r}{5}$; and by Art. 11, $So=SG+\frac{AG\times GO}{SG}=SG+\frac{2r^2}{5SG}$, and $Go=\frac{2r^2}{5SG}$.

Cor. 1. If $SG=2r$, i. e. if the sphere be suspended from a point at the distance (r) from the surface of the sphere, then $Go=\frac{2r^2}{10r}=\frac{1}{5}$th the radius of the sphere. If $SG=40$ inches, and the *radius of the sphere=one* inch, then $Go=\frac{2r^2}{5SG}=\frac{2}{5\times 40}=\frac{1}{100}$th part of an inch.

Cor. 2. Let g be the center of gravity of the *hemisphere* (AEF), then (by Exam. 3, page 142, Part. I.) $Ag=\frac{5}{8}r$. Now the value of $\frac{\frac{1}{3}\pi r^2x^3+\frac{1}{4}\pi rx^4-\frac{3}{20}\pi x^5}{sg}$, when $x=r$, $g=\frac{5}{8}r$, and $s=\frac{2}{3}\pi r^3$, is $\frac{26r}{25}$; hence, if the hemisphere be suspended by its *vertex*, $AO=\frac{26r}{25}$; $\therefore GO$ (G being the center of the sphere) $=\frac{r}{25}$; consequently the *center of oscillation* falls at the distance of $\frac{1}{25}$th radius *below its base*.



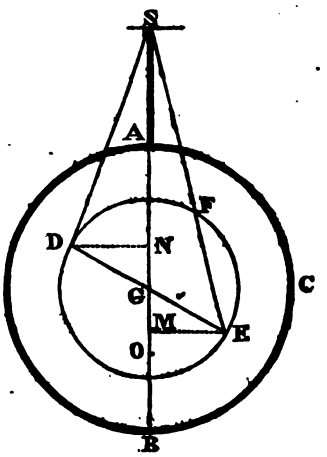
Cor. 3. Since $Ag=\frac{5}{8}r$, and AO (O being the center of oscillation of the *hemisphere*) $=\frac{26r}{25}$, we have $gO=\frac{83r}{200}$; if therefore the hemisphere be suspended from any point (S), and o be the *center of oscillation*, then $go=\frac{Ag\times gO}{Sg}=\frac{5\times 83r^2}{8\times 200Sg}=\frac{83r^2}{320Sg}$ and $So=Sg+\frac{83r^2}{320Sg}$; take SA from each side of the equation, then $Ao=Ag+\frac{83r^2}{320Sg}$; let $Ao=r$, put $\frac{5r}{8}$ for Ag, and $SA+\frac{5r}{8}$ for Sg, then $r=\frac{5r}{8}+\frac{83r^2}{320.SA+\frac{5r}{8}}$ is an equation from which we obtain $SA=\frac{r}{15}$, which gives the distance of the *point of suspension*

suspension (*S*) from the vertex of the hemisphere, when the *center of oscillation* falls in the *center of the base.*

EXAM. IV.

Let a circular plane ABC whose radius is (*r*) *vibrate in its own vertical plane; It is required to find the distance of its center of oscillation* (*O*) *from the point of suspension* (*S*).

Let *DFE* be a *concentric* circle; draw any diameter *DE*, and join *SD*, *SE*. Let $SG=d$, *DG* or $GE=x$, then $SD^2+SE^2=2SG^2+2GD^2$[a] $=2.\overline{d^2+x^2}$; the number of particles in the circumference of the circle $DFE=2\pi x$; hence "the "sum of each particle in "that circumference, "multiplied into the "square of its distance "from *S*," is $2\pi x.\overline{d^2+x^2}$, and *this sum* calculated for the *circular plane DFE* is $\int.2\pi x\dot{x}.\overline{d^2+x^2}=\pi d^2$



(a) For let fall the perpendiculars *DN*, *EM*, then, by Euc. B. 2, Prop. 12 and 13,

$SE^2=SG^2+GE^2\ (GD^2)+2SG\times GM$
$SD^2=SG^2+GD^2-2SG\times GN\ (GM)$
$\therefore SE^2+SD^2=2SG^2+2GD^2$.

$\pi d^2 x^2 + \frac{1}{2}\pi x^4$; $\therefore$ for the circular plane ABC it is $\pi d^2 r^2 + \frac{1}{2}\pi r^4$; hence $SO = \frac{\pi d^2 r^2 + \frac{1}{2}\pi r^4}{\text{circle } ABC \times SG} = \frac{\pi d^2 r^2 + \frac{1}{2}\pi r^4}{\pi d r^2}$ $= d + \frac{r^2}{2d}$.

Cor. Let the point of suspension be (at A) in the circumference, then $d = r$, and $AO = r + \frac{r}{2} = \frac{3}{2}r$. If O be made the *center of suspension*, then, by Art. 8, the *center of oscillation* will be in the circumference. Hence there are *two points* in a *circular plane*, from which if it be suspended, the distance between the point of suspension and center of oscillation will be the *same*, viz. "one in the *circumference*, and another "at the *middle point* between the center and cir- "cumference."

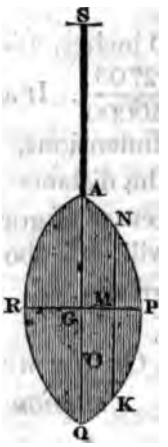
EXAM. V.

To find the center of oscillation of the solid, formed by uniting together two equal segments of a given sphere; the plane of vibration being that which passes through the common base of the two segments.

Let $APQR$ represent a section of the solid passing through the middle of it at right angles to the plane of vibration; conceive AQ to be the line into which the common base of the segments is projected, and NK the line into which a *parallel* circle

is

is projected; through the center of gravity G draw RP at right angles to AQ; and RG, PG will tend towards the *center* of the sphere from which the segments are cut. Let $PM=x$, $PG=\theta$, $MN=y$, $SG=d$, and the *radius of the given sphere* $=r$; then, from last Example, "the sum of the products "arising from multiplying each par- "ticle, contained in the circle repre- "sented by NK, into the square of its "distance from the axis of suspension "passing through S parallel to RP," will be equal to $\pi d^2y^2+\frac{1}{4}\pi y^4$, and consequently *this sum* calculated for the segment $NPQ=f\dot{}\overline{\pi d^2y^2\dot{x}+\frac{1}{4}\pi y^4\dot{x}}$; the *content* of this segment $=f\dot{}\pi y^2\dot{x}$, $\therefore SO=\dfrac{f\dot{}\overline{\pi d^2y^2\dot{x}+\frac{1}{4}\pi y^4\dot{x}}}{d\times f\dot{}\pi y^2\dot{x}}=$ (when $x=\theta$) $d+\dfrac{\frac{2}{3}\theta r^2-\frac{1}{2}\theta^2r+\frac{1}{10}\theta^3}{d.r-\frac{1}{3}\theta}$ (a). Cor.



(a) We have here calculated SO for only *half* the solid; but since the numerator and denominator of the fraction must each be doubled when the calculation is made for the *whole* solid, it is evident that the value of SO thus found will apply to both cases. The process is as follows; "Since $y^2=2rx-x^2$, "$\dfrac{f\dot{}\overline{\pi d^2y^2\dot{x}+\frac{1}{4}\pi y^4\dot{x}}}{d\times f\dot{}\pi y^2\dot{x}}=\dfrac{f\dot{}\overline{\pi d^2\dot{x}.\overline{2rx-x^2}+\frac{1}{4}\pi\dot{x}.\overline{2rx-x^2}^2}}{d\times f\dot{}\pi\dot{x}.\overline{2rx-x^2}}=$ "(dividing by πx^2) $\dfrac{d^2\overline{r-\frac{1}{2}x}+\frac{1}{4}r^2x-\frac{1}{2}rx^2+\frac{1}{10}x^3}{d.\overline{r-\frac{1}{2}x}}=$ (substi- "tuting θ for x) $d+\dfrac{\frac{2}{3}\theta r^2-\frac{1}{2}\theta^2r+\frac{1}{10}\theta^3}{d.r-\frac{1}{3}\theta}$."

Cor. Hence $GO = \frac{\frac{2}{5}\theta r^3 - \frac{1}{2}\theta^2 r + \frac{1}{20}\theta^3}{d.r - \frac{1}{4}\theta}$. I

6 inches, $\theta = \frac{1}{2}$ an inch, $d = 40$ inches, then $\frac{2703}{56000}$. If a solid therefore, of this form, and dimensions, be suspended in such a manne the distance between the point of suspensic center of gravity be 40 *inches*, the *center of osci* will fall about $\frac{1}{20}$th of an inch below the ce gravity.

LXIII.

On the method of finding the Center of cussion of a body or system of bodies.

1. Let SA be a straight line or small c moveable about an axis passing through S at right angles to the plane of its motion; and as it revolves round S, suppose it to be opposed by an obstacle placed at the distance SP parallel to that axis. At the moment this obstacle is impinged upon by the revolving body, the point P may be considered as the center of motion of a straight lever whose arms are PS, PA. In estimating the of these two arms *as opposed to each other* the point P, it is evident that the effe *each particle* p, p' &c. will be proportional t *velocity* and its *distance from* P. Now the *cities* of the particles p, p' are proportional to distances Sp, Sp' from the center of rotation hence the *forces*, with which they are impelled P at the instant when SA strikes the obstacle

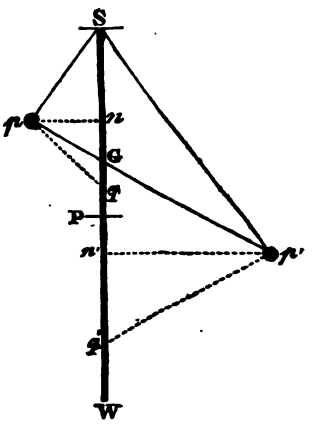
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measured by $p \times Pp \times Sp$ and $p' \times Pp' \times Sp'$ respectively. Let $Sa = a$, $Sp = b$, $Pp = x$, $Pp' = y$, then (since p, p' may be represented by $\dot{x}$, $\dot{y}$) $p \times Pp \times Sp = x\dot{x}.b - x$, and $p' \times Pp' \times Sp' = y\dot{y}.b + y$; the *sum* of the forces therefore in the *part* $Pp = f\dot{x}x\dot{x}.b - x = \frac{1}{2}bx^2 - \frac{1}{3}x^3$, and in the *part* $Pp' = f\dot{y}y\dot{y}.b + y = \frac{1}{2}by^2 + \frac{1}{3}y^3$; moreover, when $x = b$, the *effort of the arm* $PS = \frac{1}{2}b^3 - \frac{1}{3}b^3 = \frac{1}{6}b^3$; and when $y = PA = a - b$, the *effort of the arm* $PA = \frac{1}{2}b.a-b|^2 + \frac{1}{3}.a-b|^3 = \frac{1}{3}a^3 - \frac{1}{2}a^2b + \frac{1}{6}b^3$.

2. If the efforts on each side the point P be *equal*, then $\frac{1}{3}a^3 - \frac{1}{2}a^2b + \frac{1}{6}b^3 = \frac{1}{6}b^3$, and $\therefore \frac{1}{2}a^2b = \frac{1}{3}a^3$ or $b = \frac{2}{3}a$, i. e. $SP = \frac{2}{3}SA$. This point P *about which all the parts of the line or small cylinder* SA *balance each other*, when SA strikes against the obstacle, is called the *center of percussion*. If the obstacle be opposed to SA at *any other point*, these efforts do not *counteract* each other, but there is a *tendency to motion* round P; it is evident therefore that P is the point at which the revolving line or cylinder would strike the obstacle with the *greatest effect*. It may further be observed, that, since this is the point about which the parts of the line or cylinder have *no tendency to motion*, it is that at which the revolving body itself, in striking the obstacle, will be the *least liable to fracture*.

3. Suppose now p, p' to be any two particles of matter connected with each other and with their common center of gravity (G) by an inflexible rod void

void of gravity; and let another inflexible rod SGW connect their center of gravity with the axis of motion; and conceive the whole to vibrate in the same manner as the system of particles in Art. 3, page 109. Draw $pq, p'q'$ at right angles to Sp, Sp', and let fall the perpendiculars $pn, p'n'$ upon the line SGW. In the revolution of the system round S, the *velocities* of the particles p, p', in the directions $pq, p'q'$, will at every instant be proportional to their distances Sp, Sp' from S; hence their *momenta* in those directions will be measured by $p \times Sp$ and $p' \times Sp'$ respectively, and in *the directions* $pn, p'n$ by $\frac{p \times Sp \times pn}{pq}$ $\left(= \frac{p \times Sp \times Sn}{Sp} = p \times Sn\right)$ and $\frac{p' \times Sp' \times p'n}{p'q'}$ $\left(= \frac{p' \times Sp' \times Sn'}{Sp'} = p' \times Sn'\right)$. If therefore P be the *center of percussion* of the particles p, p', as referred to the line SGW, the *forces* with which these particles will urge forward the points q, q' will be measured by $p \times Sn \times Pq$ $(p \times Sn \times \overline{SP - Sq})$ and $p' \times Sn' \times Pq'$ $(p \times Sn' \times \overline{Sq' - SP})$ respectively; consequently $p \times Sn \times \overline{SP - Sq} = p' \times Sn' \times \overline{Sq' - SP}$.



But

But by sim. Δ^s, $Sq = \frac{Sp^2}{Sn}$, and $Sq' = \frac{Sp'^2}{Sn}$, $\therefore$ $p \times Sn \times SP - p \times Sp^2 = p' \times Sp'^2 - p' \times Sn' \times SP$,

or $SP = \frac{p \times Sp^2 + p' \times Sp'^2}{p \times Sn + p' \times Sn'} = \frac{p \times Sp^2 + p' \times Sp'^2}{\overline{p + p'} \times SG}$

(for $p \times Sn + p' \times Sn' = \overline{p + p'}.SG$, by the property of the *center of gravity*.)

4. In the same manner it might be shewn, that, if the system consisted of *any number* of particles p, p', p'' &c., and perpendiculars $pq, p'q', p''q''$ &c. $pn, p'n', p''n''$ &c. were drawn as in the preceding article, then their forces to produce motion round any point P would be measured by $\mp p \times Sp^2 \pm p \times Sn \times SP$, $\mp p' \times Sp'^2 \pm p \times Sn' \times SP$, $\mp p'' \times Sp''^2 \pm p'' \times Sn'' \times SP$ &c. Hence if P be the *center of percussion* (viz. that point about which the forces of the particles p, p', p'' &c. as referred to the line SGW, *balance each other*) then,

$p \times Sp^2 \pm p \times Sn \times SP + p' \times Sp'^2 \pm p' \times Sn' \times SP + p'' \times Sp''^2 \pm p'' \times Sn'' \times SP +$ &c. $= 0$ [a]; from which we have

$$SP = \frac{p \times Sp^2 + p' \times Sp'^2 + p'' \times Sp''^2 + \&c.}{p \times Sn \pm p' \times Sn' \pm p'' \times Sn'' \pm \&c.}$$

$= $(by Art. 2, p. 130, Vol. I) $\frac{p \times Sp^2 + p' \times Sp'^2 + p'' \times Sp''^2 + \&c.}{\overline{p + p' + p'' + \&c.}\ (w) \times SG}$;

from

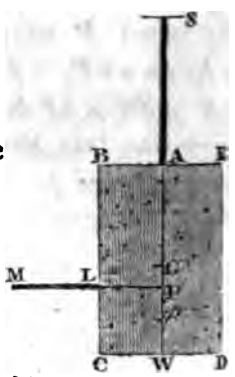
(a) By actually drawing the Figure, it may easily be shewn that the equation denoting the equilibrium will assume *this form*; in which the signs of $p \times Sp^2$, $p' \times Sp'^2$, $p'' \times Sp''^2$ &c. are all $+$, and the signs of $p \times Sn$, $p' \times Sn'$, $p'' \times Sn''$ &c. $+$ or $-$ according as the points n, n', n'' &c. lie on the *same* or *opposite* sides of S.

from which it appears, "that the distance of the "center of *percussion* from the axis of suspension "is equal to the distance of the center of *oscillation* "from the same;" the general expressions investigated in the former part of this Lecture will therefore apply to both cases.

EXAM. I.

To find the center of percussion of a cylinder (*BCDE*) *suspended from the point* (*S*), *and moving in a vertical plane passing through its axis* (*AW*).

Let h be the *height* of the cylinder, and nh the *radius of its base*; then, if p be the center of percussion when the cylinder is *suspended from A*,



$$Ap=\frac{f\sqrt{\pi y^2 x^2 \dot{x}+\frac{1}{4}\pi y^4 \dot{x}}}{sg}=\text{(since } y=nh\text{)}\frac{f\sqrt{\pi n^2 h^2 x^2 \dot{x}+\frac{1}{4}\pi n^4 h^4 \dot{x}}}{sg}$$

$$=\frac{\frac{1}{3}\pi n^2 h^2 x^3+\frac{1}{4}n^4 h^4 x}{sg}=\text{(for the whole cylinder, when } x=h,\ s=\pi n^2 h^2\text{, and } g=\tfrac{1}{2}h\text{)}\ \frac{\frac{1}{3}\pi n^2 h^5+\frac{1}{4}n^4 h^5}{\frac{1}{2}\pi n^2 h^4}=\frac{4+3n^2}{6}.h\ (a).$$

Hence $Gp=Ap-AG=Ap-\frac{1}{2}h=\frac{1+3n^2}{6}.h$. Let P be

(a) If $n=0$, then the cylinder may be considered as a right line, and $Ap=\frac{4h}{6}=\frac{2}{3}h$, which coincides with what has been shewn in Art. 1.

P be the center of percussion *when the cylinder is suspended from* S, then, by Art. 11, page 115, $SP = SG + \frac{AG \times Gp}{SG} = SG + \frac{\overline{1+3n^2}.h^2}{12SG}$; if therefore SP be taken equal to this quantity, and PM be drawn at right angles to SP, then the point L, where PM cuts the side BC of the cylinder, is the point where it must strike the obstacle.[b]

Cor. 1. If $n=1$, i.e. *if the height of cylinder be only equal to the radius of its base,* then $AP=\frac{7h}{6}$, and, consequently, P falls *below the base* at the distance of $\frac{1}{6}h$ from its center. A cylinder therefore of these relative dimensions, suspended from the *extremity of its axis,* has no center of percussion *within it;* but by Art. 8, page 113, if it be suspended from a point at the distance of $\frac{1}{6}h$ from that extremity, the center of percussion will just fall in the *center of its base.*

Cor. 2. Since $SG=SA+AG=SA+\frac{1}{2}h$, let $SA=mh$, then $SP=SG+GP=\overline{m+\frac{1}{2}}.h+\frac{\overline{1+3n^2}.h^2}{12.\overline{m+\frac{1}{2}}.h}$[c] $=\frac{12.\overline{m^2+m}+3n^2+4}{12m+6}.h$, which is a general expression for

[b] For since a force acting in the direction MP or PM at *right angles* to SP is the same at whatever point it acts at in that line, it makes no difference whether the cylinder strikes the obstacle at the point L or at the point P.

[c] For $GP=\frac{AG\times Gp}{SG}=\frac{\frac{1}{2}h.\overline{1+3n^2}.h}{6.\overline{mh+\frac{1}{2}h}}=\frac{\overline{1+3n^2}.h^2}{12.\overline{m+\frac{1}{2}}.h}$.

Exam. 3, page 121, $So=a+\frac{2r^2}{5a}=\frac{5a^2+2r^2}{5a}$, and $So'=\frac{5b^2+2R^2}{5b}$; $\therefore So\times Sg=\frac{5a^2+2r^2}{5}$, and $So'\times Sg'=\frac{5b^2+2R^2}{5}$; but by Art. 12, page 116, $SP=\frac{R^3\times So'\times Sg'+r^3\times So\times Sg}{R^3+r^3.SG}=\frac{R^3.5\overline{b^2+2R^2}+r^3.5\overline{a^2+2r^2}}{5.\overline{bR^3-ar^3}}$.

Cor. 1. If $a=0$, i. e. "if the axis of suspension "passes through the center of the smaller sphere," then $SP=\frac{R^3.5\overline{b^2+2R^2}+2r^5}{5bR^3}$; and *vice versâ*, if the larger sphere be uppermost, and the axis of suspension passes through P, and the smaller sphere be placed with its center at the distance of $\frac{R^3.5\overline{b^2+2R^2}+2r^5}{5bR^3}$ from P, then the *center of percussion* will be in that center.

Cor. 2. If $a=0$, and $b=R+r$; i. e. "if the "spheres *touch* each other, and the axis of suspension "passes through the center of the smaller sphere," then $SP=\frac{R^3.5\overline{\overline{R+r}^2+2R^2}+2r^5}{5.\overline{R+r}.R^3}$; and "if the "spheres be *equal*," (or $R=r$,) then $SP=\frac{7R}{5}$.

Cor. 3. If $a=r$, and $b=R$, i. e. "if the spheres "*touch* each other, and the axis of suspension "passes

"passes through the point of contact," then $SP = \frac{\frac{2}{5}R^2 - r^2}{\frac{1}{5}R^2 - r^2}$. If $r = 0$, $SP = \frac{2}{5}R$, which agrees with Exam. 2, page 131, for in this case "P is the center "of percussion of a sphere, whose axis of suspen- "sion passes through the extremity of one of its "diameters."

Lecture XVIII.

ON

THE CENTER OF SPONTANEOUS ROTATION.

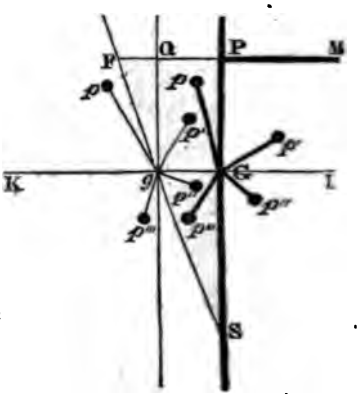
HAVING shewn the method of finding the centers of *oscillation* and *percussion* of a body or system of bodies, we proceed to the center of *spontaneous rotation.*

LXIV.

On the method of ascertaining the Center of Spontaneous Rotation of a body or system of bodies, and the angular velocity of the system round it.

1. The particles of matter p, p', p'' &c. are connected with each other and with their common center of gravity (G) by inflexible rods void of gravity. Let the motion of these particles be referred to the same plane, viz. *a plane passing through G at right angles to an axis passing also through G*; let PGS be a line drawn in that plane, and representing also an inflexible rod void

void of gravity. Suppose the axis passing through G and the rod PGS to be *moveable*, and that a force (φ) acting in direction LG at right angles to PGS was impressed upon them *at the point* G, and that this force was such as to make the center of gravity (G) move uniformly along the straight line GK; then



the *whole system* would move uniformly forward, each particle retaining its position with respect to the line PSG, which in this case would move *parallel to itself*. If the *same force* (φ) be applied in direction MP (parallel to LG) at any point (P) of that line *which is not the center of gravity*, then the line PGS will not begin to move forward parallel to itself, but will be instantly thrown into some *oblique* position FS, cutting GK in the point g. Through g draw Qg parallel to PGS, and produce MP to F. Now for a moment of time the point S may be considered as *stationary*; if therefore the angle PSF be taken indefinitely small (in which case SF approaches to equality with SP, gF with gQ, and Sg with SG), then Gg and PF will be the lines actually described in *fixt space* by the points G and P respectively in that moment of time; and since $PQ = Gg$, QF will be

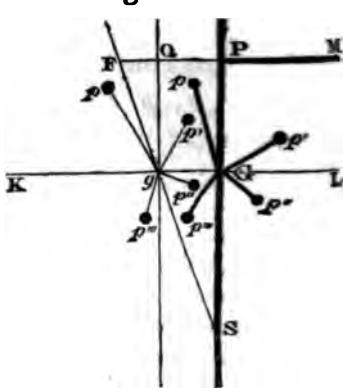
be the space described in the same time by the point (P) *round the center of gravity*[a].

2. Since PQ and QF are the spaces uniformly described in the *same time* by the points G and P when the system is first put in motion by the force (φ), they will respectively measure "the *uni-*"*form velocity* with which the center of gravity (G) "begins to move along the straight line GK" and "the *linear velocity* with which the point P begins "to revolve about G as a center;" and by the *first law of motion* the points G and P will *continue* to move with these velocities, unless the system be acted upon by some extraneous force. Now by sim. Δ^s, $PQ : QF :: Sg : qF :: SG : GP$; we are therefore arrived at this conclusion, that "if a system of par-"ticles p, p', p'' &c. connected together in the "manner here represented, be put in motion by a "force

(a) This may probably require some explanation. At the instant the force (ϕ) is impressed at P, it is evident that every point of the system, *except the center of gravity* (G), will have a tendency to *rotatory motion*; this center therefore will move forward in a *straight line*, in the same manner as if a body equal to $p+p'+p''+$&c. was placed at G, and acted upon by that force. Now since the point P moves through the space PF, whilst the center of gravity moves through the space Gg ($=PQ$), it *precedes* the center of gravity by the distance QF; but the motion of the point P is a *rotatory* motion, which, as referred to g, will be measured by the angle QgF; and as the angle PSF (and consequently its *equal* QqF) is indefinitely small, QF may be considered as a small *circular arc* described with radius gQ and center g.

" force (ϕ) acting at the point P, then their common " center of gravity (G) will move uniformly forwards " along the straight line GK parallel to the direc- " tion (MP) in which that force is impressed; " whilst the point P will be made to revolve round " the center of gravity, with a uniform velocity " which bears to *the velocity of that center along the* " *line* GK the ratio of $GP : SG$." The point S, about which the system has at every instant a *tendency* to revolve, is called the center of SPONTANEOUS ROTATION; and we now proceed to find its distance from the *center of gravity* (G).

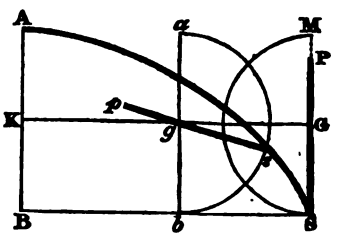
3. Since the *mass moved* along the line GK is $p+p'+p''+$ &c., and the *moving force* is (ϕ), the *accelerative force* with which the center of gravity moves will be measured by $\frac{\phi}{p+p'+p''+\text{\&c.}}$; and since the *inertia* with which the particles p, p', p'' &c. would resist the action of the force (ϕ) applied at the distance GP is $\frac{p\times pG^2+p'\times p'G^2+p''\times p''G^2+\text{\&c.}}{GP^2}$ (Note *a*, page 51), the *accelerative force* with which the point P revolves round the center of gravity (G) will be measured by $\frac{\phi\times GP^2}{p\times pG^2+p'\times p'G^2+p''\times p''G^2+\text{\&c.}}$



But

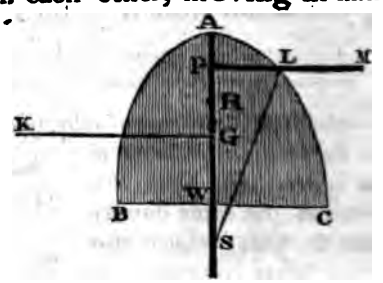
But the *velocities* of the points G and P are respectively proportional to these *accelerative forces;* hence the *velocity of the point* G : the *velocity of the point* P :: $\frac{\phi}{p+p'+p''+\&c.} : \frac{\phi \times GP^2}{p \times pG^2+p' \times p'G^2+p'' \times p''G^2+\&c.}$:: (by the last Article) $SG : GP$; hence $SG = \frac{p \times pG^2+p' \times p'G^2+p'' \times p''G^2+\&c.}{p+p'+p''+\&c.\ (w)\ GP}$. Comparing this value of SG with that of GO in Art. 6, page 112, it appears that the distance (SG) of S from the center of gravity is the same as that of the *center of oscillation* from it[a]; the distance therefore of the *center*

(a) Since the angular velocity of the *line PGS* is the same as the angular velocity of the *system*, and since S is always at the same distance from G, it is evident that during every revolution of the system the point S will describe the *common cycloid*. For with center G at the distance GS describe a circle, and let it be the *generating circle* of the cycloid AS; let bgs be the angle through which the system has revolved whilst the center of gravity has described Gg; produce sg to p, then pgs is the position of the line PGS at the end of that time, and s the situation of the *center of spontaneous rotation*. Now since the angle QgF is equal to the angle GSg (Fig. in page 138), when $gQ = Sg$, QF will be equal to Gg; i.e. "the velocity of any point in the system *whose distance from* G "*is equal to* SG, is the same as the velocity of the *center of* "*gravity*;" hence the *arc* bs is equal to Gg; but $Gg = bS$, $\therefore$ the *arc* $bs = bS$, and consequently s is a point in the cycloid AS. (For the properties of the Cycloid, see the next Lecture.)



center of spontaneous rotation (*PS*) from the point (*P*), where the force (ϕ) is impressed, is equal to the distance of the center of *oscillation* (or of *percussion*) from it, supposing *P* to be the point of *suspension*; from which it is evident that the rules which have been laid down for ascertaining the centers of *oscillation* or *percussion* of given bodies, will apply also to the finding their centers of *spontaneous rotation*.

4. In the theory of collision, as exemplified in the 5th Lecture, small *spherical* bodies were supposed to impinge upon each other, moving in lines joining their respective centers of gravity. In this case there would evidently be no tendency whatever to *rotatory* motion in the bodies after impact; but from Art. 2, it appears that when a body (*BAC*) of *any shape whatever* is impelled by a force whose direction (*ML*) does *not* pass through its center of gravity (*G*), it always proceeds after impact not only with a *rectilinear* motion along the line *GK* parallel to *ML*, but also with a *rotatory* motion about its center of gravity (*G*)[a].



The

(a) For as soon as the body is impinged upon, in any direction *MP*, at right angles to the line passing through its center of gravity, it has a *tendency* to rotatory motion round its center of *spontaneous* rotation (*S*); this rotatory motion *actually* takes place as round an

The *quantity* of this rectilinear and rotatory motion, in any given case, may be estimated in the following manner.

5. Let BAC represent a section of any solid generated by the revolution of an algebraic curve about its axis (AW); suppose this body (which may be called B) is impinged upon by another body (A) moving uniformly with the *velocity* (of a feet in 1″) in the direction MLP, and that the collision actually takes place at the point (L). Produce ML to meet the axis in P; then the effect of this collision, with respect to the rectilinear and rotatory motion of the body, must be estimated at P; and if PS be taken equal to the distance of its center of percussion from P (supposing P to be the point of suspension), S will be the center of *spontaneous rotation* at the instant that the body A impinges upon B. Take $SR=\sqrt{SP\times SG}$[b]; then supposing S to

an axis passing through the center of gravity (G) at right angles to the plane of its motion; the body (BAC) therefore, thus put in motion, is under precisely the same circumstances as the system of particles whose rectilinear and rotatory motions have been brought under consideration in Art. 1, 2, 3.

(b) For by Art. 4, p. 129, $SP=\frac{p\times Sp^2+p'\times Sp'^2+p''\times Sp''^2+\&c.}{p+p'+p''+\&c.\ SG}$,

$\therefore SP\times SG=\frac{p\times Sp^2+p'\times Sp'^2+p''\times Sp''^2+\&c.}{p+p+p''\ \&c.}=$ (by Art. 1. page 59) SR^2; hence $SR=\sqrt{SP\times SG}$; i. e. "the distance of "the center of *gyration* from the point S is a mean *proportional* "between SP and SG, whether that point (S) be the center of "*oscillation, percussion,* or *spontaneous rotation* of the solid BAC, "as referred to an axis of suspension passing through P."

S to be the center of *motion*, R will be the center of *gyration* of the body about S; $\therefore$ the *inertia* of the body (B), as referred to the *point* $P = \frac{B \times SR^2}{SP^2}$. Since also the inertia of a given body varies as the square of its distance from the center of motion, in estimating the *inertia of the body* A as connected with the *rotatory motion about* S, its mass must be reckoned equal to $\frac{A \times SL^2}{SP^2}$.

6. Let the distance of *the point where the direction of the striking body cuts the axis of the given solid* from the center of spontaneous rotation $(SP) = D$; the distance of the *center of gyration* from that center $(SR) = G$; the distance of the *center of gravity of the body* B from it $(SG) = g$, and the distance of *the point of collision* from it $(SL) = d$; then the *inertia of* A may be expressed by $\frac{Ad^2}{D^2}$, and the *inertia of* B by $\frac{BG^2}{D^2}$. Now suppose A to impinge upon B *at rest*, then the *velocity communicated to* B (Art. 6, page 158, Part. I.) $= \frac{Aa}{\text{inertia of } A + \text{inertia of } B}$ [a] $= \frac{AaD^2}{Ad^2 + BG^2}$, and this is the velocity *with which the point*

(a) By referring the common expression for the velocity communicated by a body A impinging upon another body B *at rest* $\left(\text{viz. } \frac{Aa}{A+B}\right)$ to "the theory of motion and of the mutual action "of bodies upon each other, as exemplified in this and the "preceding

point P begins to move. But the velocity *of the point* P : the velocity of the point $G :: SP : SG :: D : g$, $\therefore$ the velocity with which the *center of gravity* begins (and, since it moves uniformly, *continues*) to move along the line GK, parallel to LM, is equal to $\frac{AaDg}{Ad^2+BG^2} = \frac{AaG^2}{Ad^2+BG^2}$ (for $G^2 = Dg$) *feet in* $1''$.

7. Since the *linear* velocity of the point P is $\frac{AaD^2}{AD^2+BG^2}$, let $\pi = 3.141$ &c. then (by Art. 10, page 68) $\frac{AaD^2}{Ad^2+BG^2} : 2\pi D ::$ the angle described in $1''$ $(a) : 360°$, $\therefore a = \frac{AaD \times 360°}{2\pi . AD^2+BG^2} = \frac{AaD}{2\pi . Ad^2+BG^2}$ revolutions in $1''$ $^{(b)}$. *After* impact, therefore, the center of gravity of the body B proceeds uniformly (in a direction parallel to the direction of A's motion *before* impact) *with a velocity of* $\frac{AaG^2}{Ad^2+BG^2}$ *feet*

" preceding Lectures," it is evident that $A+B$ expresses not the sum of the *weights*, but the sum of the *inertia* of the bodies. But in the impact of bodies moving in right lines without reference to a *center of rotatory motion*, the weight and inertia are, each of them, proportional to the *quantities of matter* contained in them; so that in Lecture 5, the sum of their *inertia* is rightly expressed by the sum of the *bodies* $A+B$.

(b) We have here estimated the angular velocity of the point P about the *center of spontaneous rotation* (S); but since the angle FgQ (Fig. in page 136) is equal to the angle PSF, the angular velocity of the body or system about its *center of gravity* is equal to that of *the point* P *about* S.

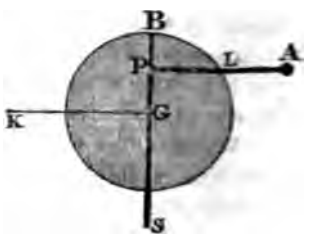
feet in 1″; *whilst the body* B *itself revolves round it at the rate of* $\frac{Aa\,D}{2\pi.Ad^2 + BG^2}$ *revolutions in* 1″.

2. If A (and consequently its *inertia*) be very small with respect to B, then Ad^2 may be neglected with respect to BG^2, and the *velocity of the center of gravity of* B after impact $= \frac{Aa\,G^2}{BG^2} = \frac{Aa}{B}$ feet in 1″ = the velocity which would be communicated by A (moving with the velocity a and) acting immediately upon the center of gravity. The *number of revolutions* performed by B in 1″ $= \frac{Aa\,D}{2\pi BG^2} = \frac{Aa\,D}{2\pi B\,D\,g} = \frac{Aa}{2\pi Bg}$

EXAM. I.

A bullet (A) *weighing* $\frac{1}{2}$ *an ounce impinges upon a sphere* (B) *with a velocity of* 100 *feet in* 1″; *its direction* ML, *when produced, bisects the radius* BG; *the weight of the sphere is* 10 *lbs, and its radius one foot; In what manner will the sphere move after impact.*

This is a case of Art. 8, where the *inertia* of A may be considered as *evanescent*; the *velocity* with which the *center of gravity* of the sphere therefore moves after impact (in the direction GK parallel to AP) $= \frac{Aa}{B} =$ (for $B = 10\,lbs$ $= 160$ *ounces*) $\frac{\frac{1}{2} \times 100}{160} = \frac{5}{16}$th of a foot in 1″. Before we can ascertain the *angular* velocity of the sphere,



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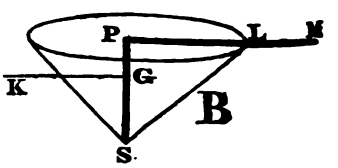
we must find the value of GS; now GS answers to Go in Exam. 3, page 121 (P being the center of *suspension*), $\therefore$ $GS=\frac{2r^2}{5PG}=$ (for $PG=\frac{1}{2}r$) $\frac{4r}{5}=$ *the distance of the center of spontaneous rotation from the center of gravity* (g); consequently $\frac{Aa}{2\pi Bg}=$ (for $2\pi=6.28$ &c.) $\frac{\frac{1}{2}\times 100}{6.28\times 160\times\frac{4}{7}}$ &c. $=\frac{50}{803}$ revolutions in 1″; a sphere therefore thus struck would proceed on with a slow *rectilinear* motion of about 4 inches in 1″, revolving at the same time about its own center at the rate of $3\frac{591}{803}$ revolutions in a *minute*.

Cor. Let $PG=\frac{1}{n}$th radius $=\frac{r}{n}$, then $g=SG=\frac{2r^2}{5PG}=\frac{2nr}{5}$; hence $\frac{Aa}{2\pi Bg}=\frac{5Aa}{4\pi nrB}$. If therefore a sphere (B) (whose *radius* $=r$) be impinged upon by a small body (A) whose *inertia* may be considered as *evanescent*, and which moves with a uniform velocity of (a) feet in 1″; and if the direction of A's motion intersects the radius of the sphere, perpendicularly, at *the distance of* $\frac{1}{n}$th *radius* from the *center* of the sphere; then the general expression for the *rectilinear* and *rotatory* motion of the sphere after impact will be $\frac{Aa}{B}$ *feet* and $\frac{5Aa}{4\pi nrB}$ *revolutions* in 1″ respectively.

EXAM. II.

A right-angled cone (*B*), *placed with its vertex downwards, and its axis* (*PS*) *perpendicular to the horizon, is struck by a small body A* (*moving, in the direction of the line ML, parallel to the horizon and passing through the center of its base*) *with a velocity of* (*a*) *feet in* 1″; *It is required to ascertain the rectilinear and rotatory motion of the cone after impact.*

This also is a case to be referred to Art. 8, so that the *velocity of the center of gravity* of the cone after impact will be $\frac{Aa}{B}$ feet in 1″. With respect to its *angular* velocity; by referring to Cor. 2, Exam. 2, page 120, it appears that when the direction of impact passes through the *center of the base* of a *right-angled* cone, the center of *spontaneous rotation* will be in the *vertex* (*S*). If therefore h = the *height* of the cone, $g = SG = \frac{3}{4}h$ [a]; $\therefore \frac{Aa}{2\pi Bg} = \frac{Aa}{2\pi B \times \frac{3}{4}h} = \frac{2Aa}{3\pi Bh}$ for *the number of revolutions in* 1″.



Cor. Suppose the blow to be struck by the hand with a small instrument, and that the *magnitude* (*Aa*) of the blow is such as would make a spherical ball, whose *diameter* is equal to the *height of the cone*, move over a space of 40 feet in 1″ by direct impact

(a) See Case 4, Exam. 1, page 139, Vol. I.

impact; then may Aa be represented by "the *mo-* "*mentum* of this ball, i.e. by $2B \times 40$[b];" hence we have $\frac{Aa}{B} = \frac{2B \times 40}{B} = 80$ feet in 1″ for the *rectilinear* motion of the cone; and $\frac{2Aa}{3\pi Bh} = \frac{4B \times 40}{3\pi Bh} = \frac{160}{3\pi h}$ for the *number of revolutions in* 1″; so that if the height of the cone be 4 *inches* (i. e. $\frac{1}{3}$d of a *foot*) it will make about 50 revolutions in a second.

EXAM. III.

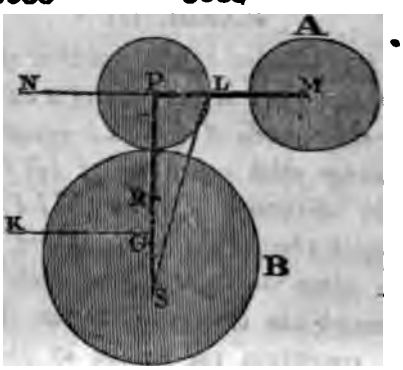
A body (B), *formed by the junction of two spheres whose radii* (R,r) *are in the ratio of* 2 : 1, *is impinged upon by another body* A (*which is equal to the lesser sphere*) *moving with a velocity of* (a) *feet in* 1″; A *moves in the direction of a line* (MLP) *joining its own center with that of the lesser sphere,* MLP *being at the same time at right angles to the line* (PRS) *passing through the centers of the two spheres; It is required to ascertain the motion of the two bodies* A *and* B *after impact.*

By referring to Cor. 2, Exam. 2, p. 133, it appears that the distance (D) of the center of *spontaneous rotation* (S) from the point of impact (P) is $\frac{197r}{60}$[c]; the distance

(b) The solid content of a *sphere* and a *cone* are respectively $\frac{4}{3}$ds, and $\frac{1}{3}$d, of their circumscribing cylinder; ∴ the solid content of the *spherical ball* $= 2B$.

(c) This is found by substituting $2r$ for R in the expression exhibiting the value of SP in that corollary.

distance PG (G being the center of gravity of the c
pound solid) $= \frac{bR^3}{R^3+r^3}$ (for $b=R+r=3r$) $\frac{8}{3}$
hence $SG=SP-PG=\frac{197r}{60}-\frac{8r}{3}=\frac{37r}{60}$, and S
(R being the center of *gyration* round S) $=SP\times S$
$=\frac{197r}{60}\times\frac{37r}{60}=\frac{7289r^2}{3600}$ (G^2); moreover, $SL^2=S$
$+PL^2=\frac{38809r^2}{3600}+r^2=\frac{42409r^2}{3600}=d^2$. Substitut



these values for D, G, d in the expressio
$\frac{AqG^2}{Ad^3+BG^3}$ and $\frac{AaD}{2\pi.Ad^3+BG^3}$ of Art. 7 (at t
same time making $A=1$, $B=8$) we have $\frac{a}{13.8}$
the number of feet described by the *center of gravi*
of B in 1″; and $\frac{a}{17\pi r}$ (very nearly) for the *numb*
of revolutions in 1″ round it.

W

(a) For by Exam. 2, page 132, $SG=\frac{bR^3-ar^3}{R^3+r^3}$; but in
case $a=0$, and P in *this* Figure corresponds to S in *Fig. p.* 1
$\therefore PG=\frac{bR^3}{R^3+r^3}$.

With respect to the *motion of A* after impact, it may be observed, that just at the *instant* the collision takes place, the bodies *A and B* have a *common velocity;* but on account of the rotatory motion of *B*, *A* only *continues* to go on with this common velocity; and this it does in the direction *LN* coincident with the direction which it had before impact[b]. Now this common velocity is the *velocity of the point P*, which, by Art. 6, is equal to $\frac{AaD^2}{Ad^2+BG^2}$; substituting therefore for A, B, D, G, d, their values as already found, we have $\frac{2a}{5}$ (very nearly) for the *velocity of A after impact.*

Cor. Since the *velocity of A* after impact is about *two fifths*, and the *velocity of B* after impact about *one thirteenth*, of the *velocity of A before impact;* and since the *common velocity* of the bodies *A* and *B* impinging upon each other by *direct* impact is *one ninth* of that velocity[c]; it appears that the velocity of

(b) It is evident that there is no cause operating to make *A deviate* from the right line in which it is moving before impact; and since, by the rotatory motion of *B*, it is immediately disengaged from *its contact with the smaller sphere*, it must necessarily proceed on in that direction with the velocity which it possesses after the act of collision has taken place.

(c) By *direct* impact is meant, "when the bodies impinge "upon each other in the line which joins their respective centers "of gravity;" in this case their *common velocity* would have been $\frac{Aa}{A+B}$ = (since $A=1, B=8$) $\frac{1}{9}a$.

of *A* after impact is *greater*, and that the velocity of *B* is *less*, than it would have been if their collision had taken place by direct impact. All this is perfectly consistent with the established laws of motion; for it is evident that the body *A*, in impinging upon *B* at *P*, has much less of *B*'s *inertia* to overcome than if it had impinged at *G*; it will therefore lose less of its *momentum* (and consequently of its *velocity*) in the *former* case than in the *latter*. On the other hand, since the body *A*, in communicating motion to *B*, expends *part of its momentum* in producing the *rotatory* motion of *B*, the *momentum* (and consequently the *velocity*) of *B*, in a *rectilinear* direction, must be *less* than what it would have been had *A* impinged upon it by *direct impact*.

We have here considered only the case of one body impinging upon another, in such a manner that "the direction of the former body intersects "a line passing through the center of gravity of the "latter at *right angles*." With respect to the *oblique* collision of bodies, the reader is referred to the 10th Section of *Mr. Atwood's Treatise on the Rectilinear Motion and Rotation of Bodies*, whose method of investigating the general expressions for the *rectilinear* and *angular* motion of the body *B* has been adopted in this Lecture.

Lecture XIX.

ON

THE VIBRATION OF PENDULUMS.

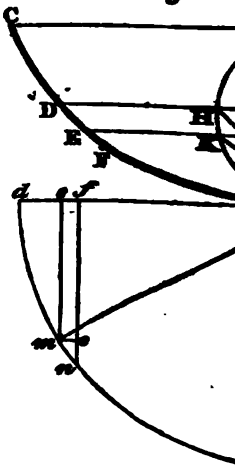
PENDULUMS are of two kinds, *simple* and *compound*. If a particle of matter, or a very small body, be attached to *one* end of a string or of an inflexible rod void of gravity, and it be made to vibrate about an axis of suspension passing through the *other*, it is called a *simple* pendulum; but if a body or system of bodies be made to vibrate about an axis of suspension passing through *any given point*, then it is called a *compound* pendulum. It is our intention in this Lecture to shew the method of finding the *times of vibration* of pendulums of both kinds[a].

(a) To avoid repetition, it may be observed, once for all, that, throughout the whole of this Lecture, the pendulums are supposed to meet with no *resistance from the air*, nor to sustain any impediment from *friction* upon the axis of suspension; the conclusions thus derived, therefore, will only be *accurately* true, when these vibrations are performed in a perfect *vacuum* round a perfectly *smooth* axis of suspension.

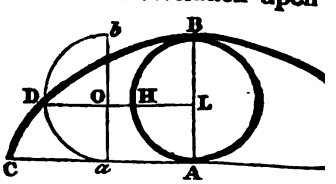
LXV.

On the method of ascertaining the ti Vibration of a simple Pendulum t the arcs of a given Cycloid, *and t very small* Circular *Arcs.*

1. *AHB* is a circle, whose diameter *AI* pendicular to the horizon; *CDB* is a *cycloi* one of whose properties[a] is this, that if, from its extremity *C*, and from any other points *D*, *C* &c. lines *CA*, *DL*, *EM* &c. be drawn *parallel to the horizon*, the whole cycloidal arc *CB* is *double* of the diameter *AB*, and the arcs *DB*, *EB*, &c. are double of the *corresponding chords HB*, *KB* &c. of the circle. Now, by Article 5, page 17, if a body begins to descend fr



(a) Let the circle *AHB* make one revolution upon *CAX*, equal to its circumference; the curve line *CDBX*, traced out by *that point of the circle which was in contact with C* when the circle began to revolve, is called a *Cycloid*. If *CX* be bisect and *AB* drawn at right angles to it, it is evident, from the

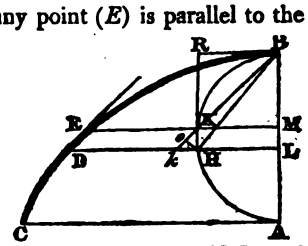


point D, its *velocity* at the *point* E will be the same as the velocity at the point M of a body falling freely through the perpendicular height LM; and its velocity at *every other point* during its descent through the cycloidal arc DEB will be the same as the velocity (of the body falling freely) at *every other corresponding point* of the line LMB. By Art. 1, p. 14, therefore, the velocity (V) of the body thus

in which the curve is generated, that it will have *similar* branches on each side AB, and that its *vertex* B will be so placed as to make its *axis* (AB) equal to the *diameter* of the generating circle. Its *properties*, as applied to the subject of this Lecture, are the following:

I. The *cycloidal ordinate* DH=*circular arc* BH. For, let bDa (Fig. in page 152) be the position of the circle when the *generating point* is at D; draw the diameter ba parallel to BA, and from D draw DHL parallel to CA; then *arc* Da=*arc* HA, $\therefore$ *sine* DO=*sine* HL, and consequently $DH=OL$; but from the mode in which the cycloid is generated, Ca=*arc* Da, and CA=*semicircle* BHA; hence $DH=OL=aA=CA-Ca$=*semicircle* BHA−*arc* HA=*arc* BH.

II. A *tangent* to the cycloid at any point (E) is parallel to the corresponding *chord* (BK) of the generating circle. Draw DHL indefinitely near to EKM; join BK, and produce it to k; let fall Ho at right angles to Kk. The indefinitely small triangle HKk is *similar* to the triangle KRB formed



by the tangents (KR, BR) to the circle at the points K, B, and is consequently *isosceles*; $\therefore$ $KH=Hk$. Now by Property I. arc $BKH=DH$, $\therefore$ $BKH-KH$ (=*arc* BK=EK)=$DH-Hk=Dk$; but since EK and Dk are *equal* and *parallel*, ED and Kk must *also* be equal and parallel; and as the *tangent* at the point E may

thus descending along the cycloidal arc will vary as $\sqrt{LM} \propto \sqrt{BL-BM} \propto \sqrt{AB \times BL - AB \times BM} \propto \sqrt{HB^2-KB^2} \propto$ (for $DB=2HB$, and $EB=2KB$) $\sqrt{DB^2-EB^2}$. Hence let Bd be drawn parallel to AC and equal to BD, and upon Bd describe the quadrant (dmN) of a circle; take Be equal to BE, and draw em at right angles to Bd; then will $em^2=Bm^2-Be^2=Bd^2-Be^2=DB^2-EB^2$, and consequently V (which varies as $\sqrt{DB^2-EB^2}$) will vary as em the *sine* of the arc dm, whose *versed sine* is de or DE *the space fallen through.*

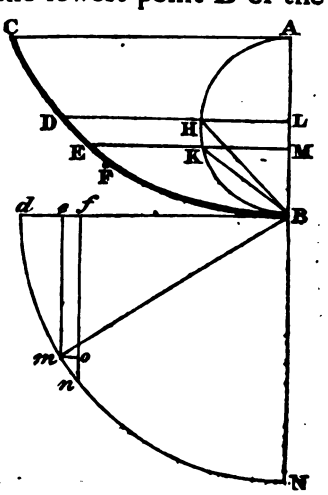
2. Let EF be an indefinitely small part of the cycloidal arc, and make ef equal to it; draw fm at right angles to Bd, and mo parallel to it. Since EF is very small, it may be considered as described with the velocity (V) at E continued uniformly, and therefore the *time* of describing EF (since $T=\frac{S}{V}$) will be represented by $\frac{EF}{V}$ or $\frac{ef}{em}$. Now since the *sine* em represents the velocity at any point E, the

be considered as coinciding with ED, it must therefore be parallel to the *chord* BK.

III. The *cycloidal arc* BE is equal to *twice the corresponding chord* BK of the generating circle. For since the triangle KHk is *isosceles*, Ho *bisects* the base Kk, $\therefore Kk$ or $ED=2Ko$; and since Ho may be considered as a *small circular arc* described with radius BH, $Ko=Bo-BK=BH-BK$; hence ED and Ko are cotemporaneous increments of the *cycloidal arc* BE and the *chord* BK; and as the arc and chord *begin* together from the point B, and the former increases by ED or Kk whilst the latter increases by $Ko=\frac{1}{2}ED$, the *arc* BE must be equal to twice the chord BK.

the whole velocity acquired in falling down DB will be represented by the *radius* BN or Bm; if therefore a body were to describe the quadrantal arc dmN with the velocity at the lowest point continued uniformly, the *time* of describing any small part (mn) of it would be represented by $\frac{mn}{Bm}$. But by similar triangles, $Bme, mno, Bm : em :: mn : mo$ or ef, $\therefore \frac{ef}{em} = \frac{mn}{Bm}$; hence the *time of describing the small cycloidal arc* EF is equal to the time of a body moving through the *corresponding small circular arc* mn with the velocity in B continued uniformly; the *whole time* of descent therefore through DEB will be equal to the time of a body's describing the *quadrantal arc* dmN with the velocity in B continued uniformly.

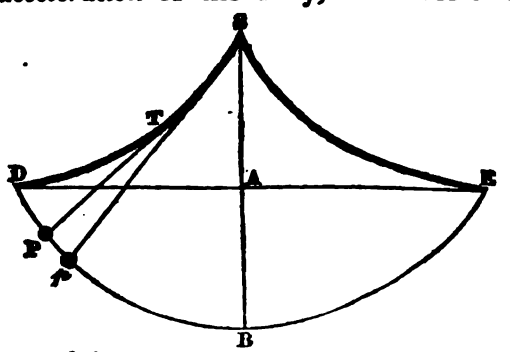
3. Now the velocity at the lowest point B of the cycloid is equal to the velocity acquired in falling down the *chord* HB, and, by Art. 4, page 5, with this velocity *continued uniformly* it would describe $2HB = BD = Bd$ in the *same* time. But by Art. 3, page 15, the time of falling down the *chord* HB = the time of falling down the *diameter* AB or the *axis* of



the

the cycloid; hence the time of descending *the cycloidal arc DB*, and the time of falling through the *axis AB of the cycloid*, are t other as the times of describing the *arc dm* the *straight line Bd* with the same uniform v i. e. as the *quadrant* of a circle to its *radius*.

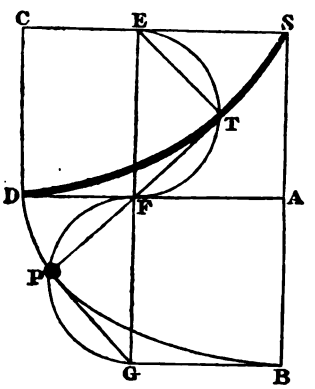
4. Hitherto we have supposed the body scend down a *plane* in the form of a cycloi the effect will evidently be the same with r to the *acceleration* of the body, whether it l



in a curve of this form by the *re-action of the* in a direction perpendicular to its surface, or *tension of a string* acting in the same dir Now a small body (*P*), suspended from the po by a string (*STP*) of the same length with of the semi-cycloids *SD*, *SE*, and made to between them (the string gradually *unwindin* the semi-cycloid *SD* as it descends to the point *B*, and *winding* round the semi-cycloid it ascends to the highest point *E*) will alw found in a cycloid *similar* and *equal* to th

semi-cycloids SD, SE; and as the string TP is always at *right angles* to the curve, it will be under precisely the same circumstances as the body descending down the plane[a]. Having descended

(a) Take any line SC, and draw SA at right angles to it; make $SC : SA ::$ *semi-circumference* of a circle : its *diameter*; and complete the parallelogram $SCDA$. Produce SA to B, making $AB=SA$; upon SC, AD describe two semi-cycloids SD, DB, the vertex of the *former* of which is in D, and of the *latter* in B; then if a body be suspended from the point S by a string whose length is equal to the semi-cycloid SD, and begins to descend from D, its place will always be in the semicycloid DB, the part TP of the string being always at right-*angles* to the cycloidal arc DB.



For through any point F draw EFG perpendicular to SC, and through B draw BG parallel to it; then $EG=SB$; on EF, FG describe the two semicircles ETF, FPG, and draw the chords TF, FP, the *former* of which, by Property II, is a *tangent* to the cycloid SD in T. Now $SE=arc\ ET$, and $SC=ETF$, $\therefore CE\ (=DF)=arc\ TF=arc\ FP$; hence the angles TEF, FGP are equal, and consequently the triangles TEF, FGP similar and equal to each other; $\therefore TF, FP$ are in the same straight line; moreover, $TP=2\ TF=$ (by Property III.) the *cycloidal arc* TD; if therefore the string TP be always equal to the *cycloidal arc* TD, i. e. if the whole string STP be equal to the semicycloid SD, P will always be found in the cycloid DB; and since the chord PG is a *tangent* to the cycloid in P, the part TP is always at *right angles* to it.

scended to the lowest point B, it will then through an arc BE equal to DB in the sa in which it descended down DB; so that th *time of one vibration* will be *twice* the t descending down DB; hence, from Art. 4, t *of one vibration* will be to the time of a body' freely down AB (i. e. down $\frac{1}{2}$ *the length of the* :: *two* quadrants of a circle : its *radius* :: t *cumference* of a circle : its *diameter*.

5. Since (by Art. 3) the *time of a vibr to the time down the axis in the same give* whatever be the point from which the pe *begins* its oscillations, all the vibrations of dulum of the *same length* will be *equal to eac* But from the difficulty of constructing plate exact form SD, SE, and from other causes, *cloidal* pendulum is of little or no *practical* In a *geometrical* point of view, however, thi of comparing the *time of a vibration* with t of falling down a space equal to $\frac{1}{2}$ *the length* string, is of considerable importance; as bles us to ascertain the times of vibration i *circular arcs*, without having recourse to a cate fluxional process. For the cycloid at th *point* B may evidently be considered as a *arc* described with the radius SB; if there body be suspended by a string whose length and vibrates in a circular arc only to a *very distance* on each side the point B, the *time*

osc

oscillation will be to the time down $\frac{1}{2}$ *the length of the string* :: *circumference* of a circle : its *diameter*.

6. Let L = length of the string or thin inflexible rod by which a small body is suspended, $\pi = 3.141$ &c., $m = 16\frac{1}{12}$ feet; then, by Art. 1. p. 33, Part I. the time of a body's falling down *half the length of the string* $= \left(\sqrt{\frac{S}{m}} =\right) \sqrt{\frac{L}{2m}}$. Hence the *time of a vibration* $(T) : \sqrt{\frac{L}{2m}} :: \pi : 1$, or $T = \sqrt{\frac{\pi^2 L}{2m}}$, which *varies as* $\sqrt{L}$; the times of vibration of pendulums of *different lengths* will consequently be to each other as the *square roots of the lengths*. Let $T = 1$, then $\sqrt{\frac{\pi^2 L}{2m}} = 1$, or $\pi^2 L = 2m$, and $L = \frac{2m}{\pi^2} = \frac{32.1666}{9.86} = 3.26$ *feet* $=$ 39.12 *inches*; if the *space fallen through from rest by gravity in* $1''$, therefore, be $16\frac{1}{12}$ feet, the length of a pendulum *which vibrates seconds* will be 39.12 inches [a].

7. If the *accelerative force be not given*, then, by Art. 2, page 3, the time of falling down $\frac{1}{2}$ the length of the string $\left(= \sqrt{\frac{S}{mF}}\right) = \sqrt{\frac{L}{2mF}}$, $\therefore T$:

(a) Let x = the *space fallen through by gravity in* $1''$, then $T = \sqrt{\frac{\pi^2 L}{2x}}$ and $T^2 = \frac{\pi^2 L}{2x}$, or $x = \frac{\pi^2 L}{2T^2}$; if therefore the *length of a pendulum which vibrates in* T'' be given, the *space fallen through*

$\sqrt{\frac{L}{2mF}} :: \pi : 1$, or $T = \sqrt{\frac{\pi^2 L}{2mF}}$, which varie

$\sqrt{\frac{L}{F}}$. Hence "the times of vibration of p
"dulums of *different lengths* acted upon by *differ*
"*accelerative forces* will vary as the square roots
"the *lengths* directly, and the square roots of
"*forces* inversely." If L be given, then $T \propto \frac{1}{\sqrt{}}$
i. e. "the times of vibration of pendulums of
"*same length* vary inversely as the *square roots*
"*the accelerative forces;*" and if T be given, t
$L \propto F$, i. e. "the lengths of pendulums vibrating
"the *same time* vary as the *forces which* acceler
"*them*[a]."

8.

through by gravity in 1′ may be found; thus, let the leng
the pendulum which vibrates *seconds* = 39.2 inches, then
(for $T=1$) $\frac{\pi^2 L}{2} = \frac{9.86 \times 39.2}{2} = 193.256$ *inches*, or 16
feet in 1′.

([a]) The two great *natural* causes which produce a vari
in the length of a pendulum which shall perform its oscilla
in *the same given time*, are, I. The *increase of gravity* a
takes place in different latitudes, as we proceed from the Eq
to the Poles. II. The *diminution of gravity which it experie*
when carried to different points above the horizon, from pl
having the same latitude. Although it would be foreign to
purpose of this Lecture, to enter into a minute investigation c
causes from which this change of gravity proceed; yet it wi
proper to make a few observations upon the *effects* produce
it, so far as relates to *the length of a pendulum oscillating se*
at different parts of the earth's surface.

1.

8. Let n = the number of vibrations performed by a pendulum in any given time (T'), and T = the time of *one* vibration, then $T' = nT$, $\therefore n = \frac{T'}{T}$; but $T = \sqrt{\frac{\pi^2 L}{2mF}}$, $\therefore n = \sqrt{\frac{2mFT'^2}{\pi^2 L}}$, which varies as $\sqrt{\frac{F}{L}}$. Hence the *number of vibrations performed in a given time* by pendulums of different lengths acted upon by different accelerative forces are *directly* as the square roots of the forces, and *inversely* as the square roots of the lengths; if therefore

I. The difference which exists in the gravity of bodies at places not having the same latitude, arises from the *centrifugal force* produced by the rotatory motion of the Earth round its axis. This force is opposed to the *force of gravity*; it is greatest at the *Equator*, and least at the *Poles*; and at intermediate points varies as the *square of the cosine of latitude*; the gravity of a body therefore will be least at the *Equator*; it will continually increase as the *latitude of the place* increases; and will be greatest at the *Poles*. It appears, from observations made upon the vibration of pendulums at different parts of the Earth's surface, and from calculations resulting from them, that if the force of gravity at the *Equator* be represented by 1.00000, at the *Poles* it will be represented by 1.00567 (See Le Place, *Exposition du Systême du Monde*, p. 250, Ed. 1808); at the intermediate latitudes of 30°, 45°, 52°, and 60°, it will be represented by 1.00141, 1.00283, 1.00357, and 1.00423 respectively; and since the lengths of pendulums vibrating in the *same time* are as the *accelerative forces*, these

therefore the *lengths* be given, the number of
lations will be directly as the *square roots*
forces; and if the *forces* be given, the numb
oscillations will be inversely as the *square r*
the lengths.

EXAM. 1.

What is the time (*T*) *of an oscillation of a*
lum whose length is 10 *feet; and what must*
length (*L*) *of a pendulum which shall oscill*
times in a minute?

numbers will represent the proportion between the le
pendulums vibrating SECONDS in these different latitud
length of the pendulum at the *Equator* being represented b

II. In estimating the effects produced by the f
gravity in this and the preceding Lectures, we h
along considered it as a *constant* force, which is on
for *very small* distances above the horizon. It has
been shewn in the Notes to Vol. I, that this force
inversely as the square of the distance from the cente
Earth; in calculating its effects therefore at any cons
distance from the surface, this *diminution* of gravity i
taken into the account. The length of the Earth's r
3970 miles; if therefore the force of gravity at the
surface be represented by *unity*, this force at the dist
1, 2, and 3 miles *above* it will be measured by the fr
$\left(\frac{3970}{3971}\right)^2$, $\left(\frac{3970}{3972}\right)^2$, $\left(\frac{3970}{3973}\right)^2$, or by the numbers .9994, .9988
respectively; which numbers (since $L \propto F$) will repres
relative lengths of pendulums oscillating *seconds* at those
distances, the length of the pendulum at the *surface* being
sented by *unity*.

The length of a pendulum which oscillates in 1″ is 39.2 inches; and, by Art. 6, the times of vibration of pendulums of different lengths, are to each other as

Taking then both these causes of variation into consideration, and referring to the pendulum whose length is 39.2 *inches* (which oscillates *seconds* at the Earth's surface in latitude 52°;) as a *standard;* the length of a pendulum that oscillates seconds *at* and for 3 *miles above* the Earth's surface at the *Equator* and the *Pole*, and for the several intermediate latitudes of 30°, 45°, 52° and 60°, may be exhibited in the following manner.

LATITUDE.	Length of a pendulum oscillating *seconds* at the Earth's surface.	One Mile above the Earth's surface.	Two Miles above the Earth's surface.	Three Miles above the Earth's surface.
At the Equator	39.060	39.036	39.013	38.997
Latitude 30°	39.115	39.091	39.068	39.052
..... 45°	39.170	39.146	39.122	39.107
..... 52°	39.200	39.176	39.152	39.137
..... 60°	39.225	39.201	39.177	39.161
At the Pole	39.281	39.257	39.233	39.218

On reviewing this Table, it appears, that "the variation in the "length of a pendulum, oscillating seconds at any point *upon* or "for *three miles above* the Earth's surface, never amounts to "$\frac{3}{10}$th of an inch;" for the *greatest* variation which can take place in this whole range is between the length of a pendulum at the *Pole* (39.281), and that of one vibrating at the distance of 3 miles above the Earth's surface at the *Equator* (38.997); and this is no more than .284 of an inch.

as the *square roots of their lengths*; hence 1'

$\sqrt{39.2} : \sqrt{120} :: 62 : 109, \quad \therefore T = \frac{109}{62}$

seconds = *time of an oscillation of a pendulu*

length is 10 *feet.*

Again, let $T = 6'' =$ time of *one* oscillatio

pendulum which makes *ten in a minut*

$T : 1'' :: \sqrt{L} : \sqrt{39.2}, \quad \therefore T^2 : 1 :: L : 39$

$L = 39.2 \times T^2 = 39.2 \times 36$ *inches* $= 117\frac{1}{2}$ *feet*

EXAM. II.

Compare the times of vibrations (T, t) *pendulums whose lengths are* L, l, *when carried distances* D, d *above the Earth's surface.*

Let $r =$ the *radius of the Earth*, then sin force of gravity varies inversely as the squ the distance from the Earth's center, the which accelerates the pendulum *whose len* (L) : the force which accelerates the pen *whose length is* $(l) :: \frac{1}{\overline{r+D}^2} : \frac{1}{\overline{r+d}^2} :: \overline{r+d}^2 : \overline{r}$

but by Art. 7, $T \propto \sqrt{\frac{L}{F}}, \quad \therefore T : t :: \sqrt{\frac{}{r}}$

$\sqrt{\frac{l}{\overline{r+D}^2}} :: \overline{r+D}.\sqrt{L} : \overline{r+d}.\sqrt{l}.$

Cor. If $L=l$, then $T:t::r+D:r+d$; i. e. the times of vibration of the *same* pendulum, when carried to different heights above the Earth's *surface*, will be to each other as the distances of those heights from its *center*.

EXAM. III.

If a pendulum at the Earth's surface vibrates (m) *times in* T''; *how must its length be altered so that it may vibrate* (n) *times in* T''?

Let $L=$ the length of the pendulum which vibrates (m) times in T'', and $L+x=$ the length of that which vibrates (n) times in T''; then, by Art. 8. (since F is given) $m:n::\frac{1}{\sqrt{L}}:\frac{1}{\sqrt{L+x}}$, $\therefore m^2:n^2::L+x:L$, or $n^2L+n^2x=m^2L$, and $x=\frac{m^2-n^2.L}{n^2}$.

Cor. Let $m=n\pm y$, where y is *very small* with respect to n, then $\frac{m^2-n^2.L}{n^2}=\frac{\pm 2ny+y^2.L}{n^2}=$ (rejecting y^2 as very small with respect to $2ny$) $\frac{\pm 2Ly}{n}$, and consequently $L+x=L\pm\frac{2Ly}{n}$, which furnishes us with a convenient theorem for ascertaining the quantity by which the pendulum of a clock must be *lengthened* or *shortened* according as

as it *gains* or *loses* a few seconds or minut
a day[a].

EXAM. IV.

A pendulum, which oscillated seconds and true time at the Earth's surface, was carried to t of a mountain, and there lost (t) seconds in an What was the height of the mountain ?

Let r = *radius of the earth*, x = *height of the tain*, T = the number of seconds in an hour c

n

(a) Suppose, for instance, that a clock gains 3 minutes in i. e. instead of performing $24 \times 60 \times 60$ or 86400 vibrat a day, it performs 86400 + 180 or 86580 vibrations in that then $n = 86400$ and $y = 180$, $\therefore \frac{2Ly}{n} = \frac{360L}{86400} = \frac{L}{240}$ although this acceleration of the pendulum indicates t length is a little less than 39.2 inches, yet it is evident finding the value of $\frac{L}{240}$, L may be assumed equal to 39.2 without any material error; hence $\frac{L}{240}$ or $\frac{39.2}{240}$ (= .16 inc the quantity by which the pendulum must be *lengthened* tc it oscillate *seconds*. In the Example here given, m (= 86 *greater* than n, and therefore y is *positive*; if the pendulu a certain number of seconds in a day, then m is *less* than consequently y is *negative*; the value of $\frac{2Ly}{n}$ must, in thi be *subtracted* from L, or the pendulum must be *sho* The *theorem* alluded to in the corollary, therefore, is " *Multiply* twice the length of the pendulum by the num " seconds gained or lost, and *divide* the result by the n " of seconds in a day; the *quotient* will give the num " inches or *parts* of an inch by which the pendulum is " lengthened or shortened."

number of vibrations in an hour at the Earth's surface, then $T-t$ will be the number of vibrations in an hour *at the top of the mountain.* By Art. 8, when the *length of the pendulum is given,* the number of vibrations in a given time are directly as the *square roots of the forces* which act upon the pendulum; hence $T : T-t :: \sqrt{\frac{1}{r^2}} : \sqrt{\frac{1}{\overline{r+x}^2}} :: \frac{1}{r} : \frac{1}{r+x}$ $:: r+x : r$, $\therefore t : T-t :: x : r$, or $x = \frac{tr}{T-t}$ [b].

Cor. If the pendulum *gains* (t) seconds in an hour before it is carried up the mountain, and it be required to find that point in the ascent where it will keep *true time,* then $T+t : T :: r+x : r$, $\therefore t : T :: x : r$, or $x = \frac{tr}{T}$.

EXAM. V.

On supposition that the pendulum which oscillates seconds in latitude 52° *is* 39.2 *inches long; Compare the spaces through which a heavy body would fall from rest in* 1″ *at the Equator and the Pole, and at the intermediate latitudes of* 30°, 45°, *and* 60°.

By

(b) If t be *very small* with respect to T, then x may be considered as equal to $\frac{tr}{T}$ without any material error. Now $T=3600''$, if therefore t be equal to 1, 2, 3, &c. seconds, then x will be equal to $\frac{r}{3600}, \frac{2r}{3600}, \frac{3r}{3600}$, &c.; and supposing r to be equal to 4000 *miles,* the heights of mountains upon which a pendulum, that oscillates *seconds* at the Earth's surface, loses 1, 2, 3, &c. seconds in an hour, will be $1\frac{1}{9}$, $2\frac{2}{9}$, $3\frac{1}{3}$, &c. miles respectively.

By Note (*a*), page 159, if L = *the length of a pendulum which oscillates seconds*, and $\pi = 3.14$

then *the space fallen through by gravity in* $1'' =$

substitute for L its value as it stands in th *column* of the Table in page 163, and 9.865 f then we have 192.663; 192.934; 193 193.476; 193.753 *inches* for the spaces des in $1''$ at the *Equator*, in latitude 30°, 45°, 60 at the *Pole*, respectively.

Cor. Since the space fallen through in $1''$ *Equator* is 192.663, and at the Pole 19 inches, the *greatest difference* between the spaces through in $1''$ at any two points upon the s of the Earth is 193.753 − 192.663 or 1.09 *in*

LXVI.

On the Times of Vibration of Compou Pendulums.

1. Since the angular velocity of a body or s of bodies moving round a given axis of suspe is the same as if all the matter of the body or s were collected into the *center of oscillation*, evident that the *time of a vibration* of a comp pendulum will be measured by the angular ve of its center of oscillation, and that the *real le*

(*a*) Since the *distance between the center of oscillati the center of gravity* of a body or system (by Art. 7, pag varies inversely as the *distance between the center of*

of the pendulum will be *the distance between that center and the point of suspension.* To find the time of a vibration therefore of a pendulum of this kind, we have only to ascertain the distance between the point of suspension and center of oscillation by the rules laid down in the preceding Lecture, and substitute it for L in the equation $T=\sqrt{\frac{\pi^2 L}{2mF}}$ of Art.

and point of suspension, it is evident that in a *particle of matter* or *small body* suspended as in the preceding Section of this Lecture, the centers of gravity and oscillation may be considered as *coincident.* Take, for instance, a sphere of 2 inches diameter, and suspend it by a string 40 inches long; then, by Cor. 1, Exam. 3, page 121, *the distance between the centers of gravity and oscillation* $=\frac{1}{100}$*th* part of an inch; suppose the diameter of this sphere to be diminished whilst the length of the string remains the same, then the distance between the centers of gravity and oscillation may become *less than any assignable quantity.*

The pendulum of a clock is not made *spherical,* but in the form of the solid, a tranverse vertical section of which is represented in Exam. 5, page 124; a solid of this form being much more commodious than that of a *sphere,* inasmuch as it suffers less resistance from the air. Suppose the dimensions of a pendulum of this kind to be such as stated in that Example; then *the distance between the centers of gravity and oscillation* is $\frac{1}{20}$*th* of an inch, the distance of the center of gravity from the point of suspension being 40 *inches;* its *whole length* therefore is 40.05 inches; *shorten* the rod by which it is suspended .85 of an inch (which may be done without introducing any material error into the expression $\frac{\frac{2}{3}\theta r^2-\frac{1}{2}\theta^2 r+\frac{1}{10}\theta r^3}{d.i.-\frac{1}{3}\theta}$, which determines the value of GO), and its length is reduced to 39.2 *inches.* By this method you would obtain a pendulum which oscillates *seconds* at the Earth's surface.

Art. 7, page 160; and the value of T thus fo
will be the time required.

2. If it be required to find the *point of suspe*
of a given body or system, so that it shall per
its oscillations *in some given time* (T), then let
the length of a simple pendulum which osci
in that given time, and (since $T=\sqrt{\frac{\pi^2 L}{2mF}}$
shall have $L=\frac{2mFT^2}{\pi^2}$ for the *distance betwe*
point of suspension and the center of oscill
Suppose G to be the *center of gravity* of the
system, and A some *given point* in it (the
vertex of a given solid, for instance), through
which if the axis of suspension passes, its
center of oscillation will be in O; then (by
Art. 11, page 115) *if S be the point of sus-*
pension, $L=SG+\frac{AG\times GO}{SG}$. Let $SG=x$,
$AG=D$, $GO=a$; then $L=x+\frac{Dd}{x}$ or
$x^2-Lx+Dd=0$, from which $x=\frac{L\pm\sqrt{L^2-}}{2}$
substitute $\frac{2mFT^2}{\pi^2}$ for L in this equation, a
determines the *distance of the point* (S) *fro*
center of gravity (G), through which if the a
suspension passes, the body or system will vi
in the given time (T)[a].

3.

(a) Since x has *two* values, it appears that there
points of the system at different distances from the ce
gravity (G), from which if the system be suspended,

o

3. Since $T=\sqrt{\frac{\pi^2 L}{2mF}}$ and *varies* as $\sqrt{\frac{L}{F}}$; supposing F to be given, T will vary as $\sqrt{L}$, and $T^2 \propto L$; to find the point therefore from which the body or system must be suspended so as to perform a vibration (at a given place) *in the least time possible*, we must make $x+\frac{Dd}{x}=L=$ a *minimum*, $\therefore \dot{x}-\frac{Dd\dot{x}}{x^2}=0$, and $x=\sqrt{Dd}$; "the body "or system will consequently vibrate in the least "time possible *when SG is a mean proportional* "*between AG and GO*[b];" moreover, if (*o*) be the center

oscillate in the given time (T). This agrees with Art. 8. p. 113; for since the center of oscillation may be made the point of suspension, and *vice versâ*, there will be two points in the system (at different distances from some *given point*) which, being made the centers of suspension, will give the *same value for L*.

(b) Thus, let Aa (a) be a *straight line, or thin cylindrical rod*; then, supposing it to be suspended from A, $AO=\frac{2}{3}a$; $\therefore GO=AO-AG=\frac{2}{3}a-\frac{1}{2}a=\frac{1}{6}a$; hence $SG=\sqrt{AG\times GO}=\sqrt{\frac{1}{2}a\times\frac{1}{6}a}=\frac{a}{\sqrt{12}}$, and $SA=AG-SG=\frac{1}{2}a-\frac{a}{\sqrt{12}}=\frac{21a}{100}$ very nearly; a rod of this kind will therefore vibrate in *the least time possible* when it is suspended from a point the distance of which from its upper extremity is *about* $\frac{21}{100}$*th of the whole length of the rod.*

In the *right-angled cone* whose vertex is A (see Cor. 1. Exam. 2. p. 120) $AO=h$, and $AG=\frac{3}{4}h$, $\therefore GO=AO-AG=\frac{1}{4}h$, and $SG=\sqrt{AG\times GO}=\sqrt{\frac{3}{16}h^2}=\frac{h\sqrt{3}}{4}$; hence $SA=AG-SG=\frac{3-\sqrt{3}.h}{4}$ equal

center of oscillation corresponding to S, then $= \frac{AG \times GO}{SG} = \frac{AG \times GO}{\sqrt{AG \times GO}} = \sqrt{AG \times GO} =$ i. e. the center of gravity (G) *bisects* So.

EXAM. I.

A sphere whose radius is 2 *feet, is suspended a point at the distance of* 10 *feet from its ce What would be the time of its vibration upon t of a mountain* 2 *miles high?*

By Art. 1, $T = \sqrt{\frac{\pi^2 L}{2mF}}$; but by Exam. 3, p $L = SG + \frac{2r^2}{5SG} = 10 + \frac{8}{50} = \frac{254}{25}$, and by Not page 162, $F = .9988$; $\therefore T = \sqrt{\frac{9.86 \times 254}{32\frac{1}{6} \times .9988 \times}}$ 1.73 seconds.

EXAM. II.

A thin cylindrical rod is first suspended at its extremities, and afterwards from a point divides it into two parts which are to each ot the ratio of $n : 1$; *Compare the times of vib* (T, t) *in the two cases.*

By Cor. 2. to Exam. 1, page 117, the dista the center of oscillation from the point of pe

$= \frac{31h}{100}$ very nearly; a right-angled cone therefore will in the *least time possible* when the axis of suspension through its own axis at the distance of *about* $\frac{31}{100}$*ths of it from the vertex.*

pension in the *former* case : distance in the *latter* :: $n+1.n^2-1 : n^3+1$; and since by Art. 6, page 159, the times of vibration of different pendulums vary as the *square roots of their lengths*, $T : t :: \sqrt{n+1.n^2-1} : \sqrt{n^3+1}$.

Cor. 1. Suppose the rod to be suspended from a point $=\frac{1}{4}$ its whole length from the upper extremity, then $n=3$, and $T : t :: \sqrt{8} : \sqrt{7}$.

Cor. 2. If $n=2$, then $T=t$, as it ought to do consistentywith the principles of Art. 8, page 113.

EXAM. III.

Let a small weight W be suspended from the extremity of a rod (SW) whose length is (L); It is required to find at what point another small weight (w) must be placed so that the pendulum shall perform its vibrations in the least time possible.



Let (G) be the *center of gravity*, and (O) the *center of oscillation* of the two weights; and let $Sw = x$, then (by Note *a*, page 117) $SO = \frac{WL^2+wx^2}{W+w.SG} = \frac{WL^2+wx^2}{WL+wx} =$ a *maximum*; hence $2wx\dot{x}.\overline{WL+wx} - w\dot{x}.\overline{WL^2+wx^2} = 0$, from which $x^2+\frac{2WLx}{w}-\frac{WL^2}{w}=0$, and $x=\frac{L}{w}\sqrt{W^2+Ww}-W$.

Cor. If $w=W$, then $x=L.\sqrt{2}-1$, and $\therefore x : L :: \sqrt{2}-1 : 1 :: 5 : 12$ very nearly.

Exam.

EXAM. IV.

Suppose a thin cylindrical rod (Aa) *to be su*
from the point (S), *and a weight* (W) *to be p*
its lower extremity (a); *What weight* (x)
placed at its upper extremity, that the time of v
of the rod may be the same both with and wit
weights.

Let O be the *center of oscillation* of the rod whe
it vibrates *without* the weights; and let $Sa=$
$L, SA=l$, then, by Exam. 1, page 117, $SO=$
$\frac{2.\overline{L^3+l^3}}{3.\overline{L^2-l^2}}$; and by the preceding example
the distance of the center of oscillation from
with the weights (W,x) *appended* will be $\frac{W^2L}{WL}$
hence, that the time of vibration *with* and *with*
weights may be the *same*, $\frac{WL^2+l^2x}{WL-lx}$ must b
to $\frac{2.\overline{L^3+l^3}}{3.\overline{L^2-l^2}}$, or $x=\frac{WL}{l}\times\frac{2.\overline{L^3+l^3}-3L.\overline{L^2}}{2.\overline{L^3+l^3}+3l.\overline{L^2}}$

(a) Since W and x lie on *different sides* of S, $\overline{W+x}\times$
be equal to $WL-lx$.

END OF PART III.

AN

INTRODUCTION

TO THE

STUDY OF NATURAL PHILOSOPHY.

PART IV.

ON THE MOTION OF BODIES BY VARIABLE FORCES.

Lecture XX.

ON THE RECTILINEAR MOTION OF BODIES BY VARIABLE FORCES; AND ON THE VIBRATION OF CORDS.

In the preceding part of this volume, the motion of a body or system of bodies has in every case been supposed to be produced by the action of *constant* forces; the relation of the *space*, *time*, and *velocity*, therefore, could readily be obtained from the general expressions in Art. 2, page 3. If the force *varies* at every instant, these expressions are no longer applicable; and that relation must be ascertained by a *fluxional* process, the nature of which we now proceed to explain.

LXVII.

On the method of investigating the g expression for the velocity and time body moving through any given spa the action of a variable force.

1. Suppose a body to be made to descri space AS by the action of a force, the inten which *varies* at every instant. Let the *space* over which the body has moved in the *time* $(t) = x$, and let the velocity acquired at the end of that time $= v$. Let F = the magnitude of the force acting at the point D, $m = 16\frac{1}{12}$ feet; then (by Art. 2, page 3), if a body moves through any space (x) by the action of this force continued *constant*, the relation between F, x and the *velocity* (v) will be expressed by the equation $v^2 = 4mFx$, the *fluxion* of which is $2mF\dot{x}$. Take d indefinitely near to D, so $Dd = \dot{x}$; then, since the force may be consid *constant* through the very small space Dd, this equation (viz. $v\dot{v} = 2mF\dot{x}$) will express the re between F, x, and v whilst the body moves th that small space[a]; if therefore the value of each point were assigned in *terms of* x, the g relation between F, x, and v, when the bo moved through any space x, would be express the equation "*fluent of* $v\dot{v}$ = *fluent of* $2mF\dot{x}$," or " $f.2mF\dot{x}$."

(a) For the body moves through the small space D precisely the *same* circumstances, whether the velocity (v) b rated by the action of the *constant* force F, or by the acti *variable* force which (when the body is at the point D) is *eq*

2. Since the velocity through the indefinitely small space Dd may be considered as *uniform*, the *time* of describing it will be represented by $\frac{Dd}{v}$ or $\frac{\dot{x}}{v}$. Suppose therefore that $v=\varphi x$ [b], we shall have $i=\frac{\dot{x}}{\varphi x}$, and $t=f\!:\frac{\dot{x}}{\varphi x}$. Thus, both the *velocity* acquired and the *time* of the body's motion through any *space* AD will be ascertained in all cases, "when the fluents of $2mF\dot{x}$ and $\frac{\dot{x}}{\varphi x}$ can be found." If $SD=x$, then (since the body moves in direction AD) x decreases as v increases, therefore the *fluxions* of x and v will have *different signs*, and the time and velocity will be determined from the equations "$\frac{v^2}{2}=f\!:-2mF\dot{x}$," and "$t=f\!:\frac{-\dot{x}}{\varphi x}$."

LXVIII.

On the variation of the accelerative force of a body descending (as P in Lectures XIV. *and* XV.*) by means of a rope whose weight is taken into consideration.*

In the different machines adverted to in Lectures XIV. and XV. it is evident that the *moving force* of *P as it descends* is increased at every instant by *the variable weight of that part of the rope by which it is suspended*, and that the *inertia* of the mass moved is increased

(b) By ϕx is meant any *function* of x, i. e. "an expression "involving only x and known quantities," such as, "a^2-x^2," "$a+bx$," "$\sqrt{a^2+x^2}$," "$px+qx^2-rx^3$," &c. &c.

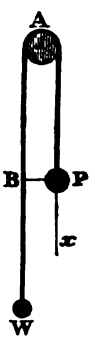
increased by *the whole weight of the rope*. If fore the weight of the rope be so considera to render it necessary that it should be take the account, the expressions for the *accel force of* P must be modified in the foll manner.

1. Let us first take the case of "a weight ("scending by means of a rope going over a "whose radius is (r), and at the same time cc "nicating a rotatory motion to a body or sys "bodies whose weight is (W), and the dist "whose center of gyration from the axis of r "is (D)." Let the rope be of uniform thi and let its whole *length* $= l$, its *weight* $= w'$, a = that part of the rope which is unwound P *begins to descend*; then the *weight* of this the rope $= \frac{w'a}{l}$; and if x = the space through w descends in any time, then $\frac{w'x}{l}$ will be the *we* such part of the rope as will be unwound time. Hence the *moving force* of P after descended through the space $x = P + \frac{w'a}{l} + \frac{w'}{l}$ *mass moved* $= P + W$ *acting at the distance* (*the axis of rotation* $+ w' =$ (by Art. 2, p $P + \frac{WD^2}{r^2} + w'$; $\therefore$ the *accelerative force of*

$$= \frac{P + \frac{w'a}{l} + \frac{w'x}{l}}{P + \frac{WD^2}{r^2} + w'} = \frac{r^2 . \overline{Pl + w'a + w'x}}{l . \overline{Pr^2 + WD^2 + w'r^2}} .$$

2. If $P=0$, i. e, "if this rotatory motion of the "body or system be produced merely by the weight "of a *very heavy rope* or a *perfectly flexible chain* "of uniform thickness," then $F=\frac{r^2w'.\overline{a+x}}{l.WD^2+w'r^2}$; and this is the force with which every point of the rope or chain keeps continually descending till the whole is unwound.

3. Let us next consider what change is produced in the accelerative force by taking the weight of the rope into consideration, "when a weight (P) draws "up another weight (W) over a *fixt pulley*." Let BW (=the difference of AP and AW before P begins to descend) $=a$, the *length* of the rope $=l$, its *weight* $=w'$, and the weight of the *pulley* $=w$; then the weight of the part $BW=\frac{w'a}{l}$, and the *moving force* with which P *begins to descend* $=P-W-\frac{w'a}{l}$. Whilst



P *descends* through any space (x), W will *ascend* through the same space; the *difference* of the ropes AP, AW will therefore be increased or diminished by a part $=2x$, the *weight* of which is $\frac{2w'x}{l}$; by this weight the moving force of P is increased, $\therefore$ when P has descended through the space (x) the *whole moving force* will be

$P-W$

$P-W-\frac{w'a}{l}+\frac{2w'x}{l}$; by Note *a*, page 79, the *in* *of the pulley* $=\frac{1}{2}w$; hence the *mass moved* $=P+$ $\frac{1}{2}w+w'$, $\therefore$ the *accelerative force of* $P\,(=$

$$\frac{P-W-\frac{w'a}{l}+\frac{2w'x}{l}}{P+W+\frac{1}{2}w+w'}=\frac{Pl-Wl-w'a+2w'x}{l.P+W+\frac{1}{2}w+w'}.$$

4. Suppose no weights to be appended, b *chain* $BACD$ to pass freely over the pulley Before the chain begins to descend let $CD=a$, then the *moving force* with which it *begins* to descend $=\frac{w'a}{l}$; and when it has descended through any space (x), this force will be increased by $\frac{2w'x}{l}$; the *mass moved* is $\frac{1}{2}w+w'$; hence the *accelerative force* with which the chain descends $=$



$$\frac{\frac{w'a}{l}+\frac{2w'x}{l}}{\frac{1}{2}w+w'}=\frac{w'a+2w'x}{l.\frac{1}{2}w+w'}.$$

EXAM. I.

In the different cases of Rotatory Motio Lecture XIV, *suppose P to begin to descend the edge of the wheel, what velocity will it ha quired when it has descended through the whole l of the rope; and what will be the whole time descent?*

1

1. The value of a in the expression for the *accelerative* force in Art. 1, p. 178, is in this case $= 0$, $\therefore F = \frac{r^2 . \overline{Pl + w'x}}{l . Pr^2 + WD^2 + w'r^2}$; hence (Art. 1. p. 176), $v\dot{v} = 2mF\dot{x}$ $= \frac{2mr^2 . \overline{Pl\dot{x} + wx\dot{x}}}{l . \overline{Pr^2 + WD^2 + w'r^2}}$, and $v^2 = \frac{4mr^2 . \overline{Plx + \frac{1}{2}w'x^2}}{l . \overline{Pr^2 + WD^2 + w'r^2}}$ (a) $= (\text{when } x = l)\ \frac{4mlr^2 . \overline{P + \frac{1}{2}w'}}{Pr^2 + WD^2 + w'r^2}$; consequently the *velocity of P just at the instant the whole rope is unwound* $= \frac{2r\sqrt{ml . \overline{P + \frac{1}{2}w'}}}{\sqrt{Pr^2 + WD^2 + w'r^2}}$.

2. Let $\frac{4mr^2}{l . \overline{Pr^2 + WD^2 + w'r^2}} = \frac{2n}{w'^2}$, then $v^2 = \frac{n}{w'} . \overline{\frac{2Plx}{w'} + x^2}$, and $v = \sqrt{\frac{n}{w'}}\sqrt{\frac{2Plx}{w'} + x^2}$; hence $\dot{t} = \frac{\dot{x}}{v} = \sqrt{\frac{w'}{n}} \times \frac{\dot{x}}{\sqrt{\frac{2Plx}{w'} + x^2}}$ (Art. 2, page 177),

$$\therefore t = \sqrt{\frac{w'}{n}} \times \text{hyp. log.} \frac{\frac{Pl}{w'} + x + \sqrt{\frac{2Plx}{w'} + x^2}}{\frac{Pl}{w'}}$$

$$= \sqrt{\frac{w'}{n}} \times \text{h. l.} \frac{Pl + w'x + \sqrt{2Plw'x + w'^2x^2}}{Pl}$$

$$= (\text{when } x = l)\ \sqrt{\frac{w'}{n}} \times \text{h. l.} \frac{P + w' + \sqrt{2Pw' + w'^2}}{P},$$

seconds, for *the whole time of the unwinding of the rope.* Ex.

(a) Having ascertained the velocity of P after it has descended through any space (x), the *linear* and *angular* velocities of the body or system may be found according to the rules laid down in Art. 9, 10, pp. 67, 68.

EXAM. II.

A perfectly flexible chain, whose length is (l) and weight w', is wound round a cylinder whose weight is W and radius r; a part (a) of the chain (just sufficient to put the cylinder in motion) is unwound, and it is then left to descend by its own weight; In what time will the chain leave the cylinder?

This is a case of Art. 2, p. 170, where $F=\frac{r^2w'.a+x}{l.WD^2+w'r^2}$;
but since in the cylinder $D=r\sqrt{\frac{1}{2}}$, $WD^2=\frac{1}{2}Wr^2$,
$\therefore F=\frac{w'.a+x}{l.\frac{1}{2}W+w'}$; hence $r\dot{v}=2mF\dot{x}=\frac{2mw'.a\dot{x}+x\dot{x}}{l.\frac{1}{2}W+w'}$,
and $v^2=\frac{2mw'.2ax+x^2}{l.\frac{1}{2}W+w'}$, or $v=\sqrt{\frac{2mw'}{l.\frac{1}{2}W+w'}}\times\sqrt{2ax+x^2}$
hence $t=\frac{\dot{x}}{v}=\sqrt{\frac{l.\frac{1}{2}W+w'}{2mw'}}\times\frac{\dot{x}}{\sqrt{2ax+x^2}}$,

$$\therefore t=\sqrt{\frac{l.\frac{1}{2}W+w}{2mw'}}\times\text{hyp. log.}\frac{a+x+\sqrt{2ax+x^2}}{a},$$

$$=(\text{when } x=l-a)\sqrt{\frac{l.\frac{1}{2}W+w'}{2mw'}}\times\text{h. l.}\frac{l+\sqrt{l^2-a^2}}{a}$$

seconds, for the time in which the chain will unwind itself.

EXAM. III.

Two weights P and W are connected together by a rope whose length is (l) and weight (w'); P draws up W through the whole length of the rope, over a fixt pulley whose weight is (w); In what time will this be effected?

In

In this case (since P begins to descend from the pulley A) a will be equal to l; the expression, therefore, in Art. 3, p. 179, becomes $\frac{Pl-Wl-w'l+2w'x}{l.P+W+\frac{1}{2}w+w'}$.

To simplify the operation,

Let $Pl-Wl-w'l = nw'$,

and $l.P+W+\frac{1}{2}w+w' = qw'$,

then $F = \frac{nw'+2w'x}{qw'} = \frac{n+2x}{q}$;

hence $v\dot{v} = 2mF\dot{x} = \frac{2m.n\dot{x}+2x\dot{x}}{q}$,

and $v^2 = \frac{4m.\overline{nx+x^2}}{q}$;

$$\therefore\ t = \frac{\dot{x}}{v} = \sqrt{\frac{q}{4m}} \times \frac{\dot{x}}{\sqrt{nx+x^2}},$$

and $t = \sqrt{\frac{q}{4m}} \times \text{h.l.} \frac{\frac{1}{2}n+x+\sqrt{nx+x^2}}{\frac{1}{2}n}$

$= (\text{when } x=l)\ \sqrt{\frac{q}{4m}} \times \text{h.l.} \frac{\frac{1}{2}n+l+\sqrt{nl+l^2}}{\frac{1}{2}n}$ seconds

for *the whole time of P's descent.*

EXAM. IV.

A perfectly flexible chain whose length is (l) *and weight* (w') *is suspended over a fixt pulley whose weight is* (w); *one end hangs* (a) *inches below the other when it begins to descend*; *How long will the chain be in passing over the pulley?*

By Art. 4, p. 180, $F = \frac{w'.\overline{a+2x}}{l.\frac{1}{2}w+w'}$,

$$\therefore\ v\dot{v} = 2mF\dot{x} = \frac{2mw'.\overline{a\dot{x}\times 2x\dot{x}}}{l.\frac{1}{2}w+w'};$$

and

$$\text{and } v^2 = \frac{4\pi a^2 \cdot cx - x^2}{\frac{1}{2}\pi - a^2},$$

$$\text{hence } \frac{s}{t} = \sqrt{\frac{\frac{1}{2}\pi - a^2}{4\pi a^2}} \cdot \frac{x}{\sqrt{cx - x^2}},$$

$$\text{and } t = \sqrt{\frac{\frac{1}{2}\pi - a^2}{4\pi a^2}} \cdot \text{h.l.} \frac{\frac{1}{2}c - x - \sqrt{cx - x^2}}{\frac{1}{2}a}$$

$$= \left(\text{when } x = \frac{c-c}{2}\right) \sqrt{\frac{\frac{1}{2}\pi - a^2}{4\pi a^2}} \times \text{h.l.} \frac{c - \sqrt{c^2 - c^2}}{c}$$

seconds, for the whole time in which the chain is passing over the pulley.

Cor. If the weight of the pulley be too inconsiderable to be taken into the account, then $a = 0$,

$$\therefore \quad t = \sqrt{\frac{i}{4\pi}} \times \text{h.l.} \frac{l + \sqrt{l^2 - a^2}}{a};$$

the value of t, therefore, is in this case independent of the weight of the chain. Let its length be 300 inches, and let $a = 4$ inches; then (since $\pi = 19\frac{1}{7}$ feet $= 163$ inches) we have

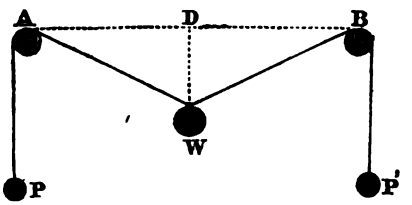
$$t = \sqrt{\frac{300}{163}} \times \text{h.l.} \frac{300 - \sqrt{89954}}{4}$$
$$= .923 \times 2.3025 \times \log. 149.99$$
$$= 3.1203 \text{ seconds.}$$

LXVIII.

On the variation of the accelerative force when one weight acts obliquely upon another over a fixt pulley, the weight of the rope and the inertia of the pulley not being taken into the consideration.

1. Let

1. Let the weight W descend against the action of the two *equal* weights P, P' appended to it by a string going over the fixed pulleys A, B. On account of the equality of the weights P, P', it is evident that W will be *similarly* situated with respect to



the pulleys; if therefore WD be drawn perpendicular to AB, it will *bisect* it; let $AD=a$, $DW=x$. By the resolution of forces, the moving force of P as opposed to W : its weight (P) :: DW : AW :: x : $\sqrt{a^2+x^2}$; hence the moving force of $P=\frac{Px}{\sqrt{a^2+x^2}}$; for the same reason the moving force of $P'=\frac{P'x}{\sqrt{a^2+x^2}}$; since therefore $P+P'=2P$, the moving force of the two weights P, P' as opposed to $W=\frac{2P}{\sqrt{a^2+x^2}}$. The moving force of W is its *own weight*, hence the *moving force with which W actually descends* $=W-\frac{2Px}{\sqrt{a^2+x^2}}$; and as the *mass moved* $=2P+W$, the *accelerative force* $=\frac{1}{2P+W}.W-\frac{2Px}{\sqrt{a^2+x^2}}$.

2. Let

2. Let v = the *velocity* of W at the distance DW (x) from the horizontal line AB, supposing that it descended along that line by the action of this accelerative force (F), and that it suffered no *alteration* of its velocity from the circumstances under which it is placed by its connection with the weights P, P';

then $v\dot{v} = 2mF\dot{x} = \frac{2m}{2P+W}.\overline{W\dot{x} - \frac{2Px\dot{x}}{\sqrt{a^2+x^2}}}$,

$$\therefore v^2 = \frac{4m}{2P+W}.\overline{Wx - 2P\sqrt{a^2+x^2}} + \text{Cor}^n.;$$

and if b = the distance of W from the line AB when the bodies *begin* to move, then

$$0 = \frac{4m}{2P+W}.\overline{Wb - 2P\sqrt{a^2+b^2}} + \text{Cor.}$$

$\therefore$ the fluent properly corrected is,

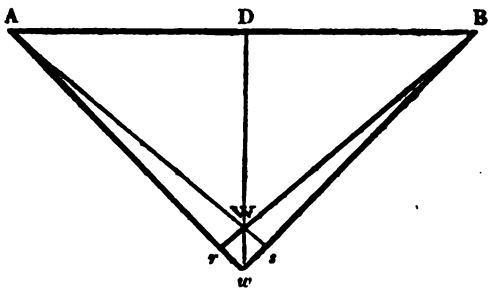
$$v^2 = \frac{4m}{2P+W}\overline{W.\overline{x-b} - 2P.\sqrt{a^2+x^2} - \sqrt{a^2+b^2}};$$

put $\sqrt{a^2+b^2} = c$, and $\sqrt{a^2+x^2}\,(AW) = y$; then

$$v = \sqrt{\frac{4mW.\overline{x-b} - 8Pm.\overline{y-c}}{2P+W}}.$$

3. We have thus obtained an expression for the velocity of W, supposing it to descend along the line DW by the action of a force represented by $\frac{1}{2P+W}\overline{W - \frac{2Px}{\sqrt{a^2+x^2}}}$; but since this is also the force by which P or P' *ascend* in a perpendicular direction, the *velocity* with which they ascend will also be expressed by the

the quantity $\sqrt{\frac{4mW.x-b-2Pm.y-c}{2P+W}}$. Now it appears that the velocity of W at every instant of its descent is *greater* than the velocity of P or P', in the ratio of $AW : DW$. For let AWB, AwB, be two positions of W indefinitely near to each other, and let fall the perpendiculars Wr, Ws upon Aw, Bw,



then Ww, rw, sw, are the *contemporaneous* increments of DW, AW, BW respectively. Hence the velocity with which W actually descends : velocity with which P or P' ascends (v) :: $Ww : rw$ or sw :: (by sim. Δ^s) $AW : DW :: y : x$; $\therefore$ the actual velocity of W at the distance DW from the horizontal line $AB = \frac{vy}{x} = y\sqrt{\frac{4mW.x-b-8Pm.y-c}{2P+W.x^2}}$.

4. Let $\frac{W}{2P} = n$, then $4mW = 8Pmn$; and $2P + W = 2P + 2Pn = 2P.\overline{n+1}$; this expression for the velocity of W may therefore be transformed into

$y\sqrt{}$

$$y\sqrt{\frac{8Pmn.\overline{x-b}-8Pm.\overline{y-c}}{2P.n+1.x^2}}$$

$$=y\sqrt{\frac{4mn.\overline{x-b}-4m.\overline{y-c}}{n+1.x^2}}$$; which

comparison will be found to agree with the
pression for the velocity of the descending
as it stands in Cor. 1. Prob. III. page 131
Thomas Simpson's Miscellaneous Tracts, Ed. 175

5. This expression for the velocity of W bec
equal to 0, when

$$4mn.\overline{x-b}=4m.\overline{y-c}$$

$$\text{or } n.\overline{x-b}=y-c$$

$$\therefore n.\overline{x-b}+c=y=\sqrt{a^2+x^2}=\sqrt{c^2-}$$

and $n^2.\overline{x-b}^2+2nc.\overline{x-b}=x^2-b^2$.

Hence $n^2.\overline{x-b}+2nc=x+b$

$$\text{or } \overline{1-n^2}.x=2nc-\overline{1+n^2}.b$$

$$\therefore x=\frac{2nc-\overline{1+n^2}.b}{1-n^2}$$; if t

fore the value of n be such that this expression
is a real and positive quantity, then the weight W
descend to the distance $\frac{2nc-\overline{1+n^2}.b}{1-n^2}$ from the
AB, where its velocity will *again* become
to *nothing*. If $b=0$ (in which case $c=a$),
$x=\frac{2na}{1-n^2}=\frac{2n}{1-n^2}.AD$, which gives the lowest p
at which W will arrive when *it begins to des*
from the point D.

6. If $n=1$, or $W=2P$, then the value of x is *infinite*, which shews that when the weight W is equal to the sum of the two weights P, P', there will be no limit to its descent.

7. If n be *greater than one*, or W greater than $2P$, then the value of x becomes *negative*, in which case the problem is impossible.

8. It is therefore only when n is *less than one*, or W less than $2P$, that there will be *two* points at which the velocity of W is equal to 0, and these will be at the distances b and $\frac{2nc-\sqrt{1+n^2}.b}{1-n^2}$ from the line AB. Somewhere between those two points, therefore, its velocity will be *maximum*.

9. Now is evident that the velocity of W will be a maximum, when the velocity of P or P' is a maximum; by Art. 2,

$$v^2=\frac{4m}{2P+W}.\sqrt{W.x-b-2P.\sqrt{a^2+x^2}-\sqrt{a^2+b^2}},$$

which is a *maximum* when $W\dot{x}-\frac{2Px\dot{x}}{\sqrt{a^2+x^2}}=0$, or $W=\frac{2Px}{\sqrt{a^2+x^2}}$. But since $W-\frac{2Px}{\sqrt{a^2+x^2}}$ is the expression for the *moving force*, this force will be equal to 0, when $W=\frac{2Px}{\sqrt{a^2+x^2}}$. Hence the velocity

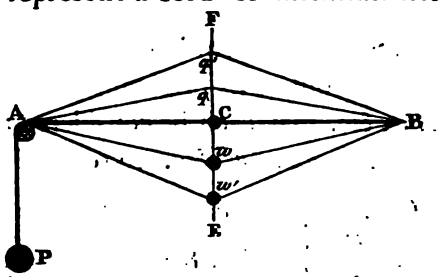
velocity is a maximum at the point of *equili* of the three bodies. That point must the always lie *between* the distances b and $\frac{2nc-1-}{1-n}$ from the line AB.

10. From which it follows, I. That if the ori position of the bodies be such that $W=\frac{2P}{\sqrt{a^2}}$ or $W : 2P :: x : \sqrt{a^2+x^2} :: DW : AW$ motion would ensue, but the system would re in *equilibrio*. II. If the distance b is such W begins to move from a point which is *abov* point of equilibrium, then it will *descend* to distance $\frac{2nc-\sqrt{1+n^2}.b}{1-n^2}$ before its velocity is stroyed; from this point it will *ascend* till it c to the distance b, from which it will beg *descend* again, and so go on ascending and scending continually. III. Lastly, if W b to move from a point *below* the point of equilib (whose distance from AB is b), then it will a till it comes to the distance $\frac{2nc-\sqrt{1+n^2}.b}{1-n^2}$, then descend again to the distance b, and s on ascending and descending as before.

LXIX.

On the Vibration of Cords.

1. Let ACB represent a cord of *inconsiderable weight*, fixed at the point B, but going over a small pulley at A; and suppose a small weight (w) to be attached to the middle point (C) of the cord. Let the cord be brought into the successive positions AwB, $Aw'B$, &c. by the action of some force in the direction CE perpendicular to AB, and let P be the weight which keeps that force in equilibrio.[b] Complete the parallelograms $AqBw$, $Aq'Bw'$, &c. then, when the cord is in the position AwB, the three forces which keep the weight (w) in equilibrio are represented in quantity and direction by wA, wq and wB, of which wA represents the *weight* P, wq *the force acting in the direction* CE, and wB the *reaction of the hook* B; hence the *force in direction* CE $(\varphi) : P :: wq : wA$, $\therefore \varphi = \frac{P \times wq}{wA}$; for the same reason, when the cord is in the position $Aw'B$, the *force in direction* CE $(\varphi') = \frac{P \times w'q'}{w'A}$; &c, &c.; consequently the force which would keep the weights



(b) i. e. let P represent the *tension* of the cord, which may be considered as the *same* in all the positions AwB, $Aw'B$, &c. without any material error.

weights P, w in equilibrio in the successive posit
AwB, $Aw'B$, &c, is a *variable one*; being 0 a
point C, and increasing continually from C towar

2. Suppose the cord to be brought only to a very s
distance out of its original position ACB, then
Aw', &c. may each be considered as equal to $\frac{1}{2}$.
and since qw, $q'w'$, &c. are respectively equal to
$2Cw'$, &c. we have in this case $\varphi = \frac{2P \times}{\frac{1}{2}A}$
$\varphi' = \frac{2P \times Cw'}{\frac{1}{2}AB}$ &c. &c.; i. e. if any variable dist
(Cw) from the point C be called x, and Al
then the force which keeps the weights P,
equilibrio at that point may be expressed by
$= \frac{4Px}{l}$. Let us now suppose the action of
force to cease, then the weight (w) will be d
towards the point C by a *moving force* which at
point of the line Cl is equal to $\frac{4Px}{l}$, and as
weight moved is (w), the *accelerative force*
which (w) is drawn towards l is equal to $\frac{4Px}{wl}$ (

3. Let E be the point to which (w) is drawn
the line AB before the action of the force (φ) ce
and let $CE = a$; then, to find the *velocity* with w
the weight (w) approaches the point C, we
$v\dot{v} = -2mF\dot{x} = \frac{-8mPx\dot{x}}{wl}$, or $v^2 = \frac{-8mPx^2}{wl} +$
for the velocity at the point w; but when
$x = a$, $\therefore v^2 = \frac{8mP}{wl} \cdot \overline{a^2 - x^2}$ when the fluen
pro

properly corrected. Hence $v = \sqrt{\frac{8mP}{wl}} \times \sqrt{a^2 - x^2}$ $= (\text{when } x = 0) \sqrt{\frac{8mP}{wl}} \times a$ for *the velocity of the point* C; and with the velocity thus acquired the weight (w) would go on to describe $CF = CE$ (on the *other side* of AB) before its velocity in the direction EF is destroyed.

4. To find the *time* in which (w) moves through the space EF, we have,

$$\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{wl}{8mP}} \times \frac{-\dot{x}}{\sqrt{a^2 - x^2}},$$

$$\therefore t = \sqrt{\frac{wl}{8mP}} \times \textit{cir. arc, whose rad. is } 1, \textit{ and cos.} = \frac{x}{a};$$

and when $x = 0$,

$$t = \sqrt{\frac{wl}{8mP}} \times \textit{quadrant whose radius } 1$$

$$= (\text{if } \pi = 3.141 \text{ \&c.}) \sqrt{\frac{wl}{8mP}} \times \frac{\pi}{2} \textit{ seconds}$$

for the time of describing EC; consequently the *time of describing* EF $(= 2t) = \sqrt{\frac{wl}{8mP}} \times \pi$ seconds. It may here be observed, that the quantity (a) does not enter into the expression for the value of t; if therefore (w) had begun to move from the points w, w', &c. then the times of describing qw, $q'w'$, &c. would have been the same as the *time of describing* EF; from whatever point, therefore, the weight (w) begins to vibrate (provided that

the whole space through which it vibrates be small) the time of *one vibration* will be equ $\sqrt{\frac{wl}{8mP}} \times \pi$ seconds; and if n = the numb vibrations in 1″, then $nt=1$, or $n=\frac{1}{t}=\sqrt{\frac{8mP}{wl}}$

5. Let us now suppose ACB to be an e cord, whose *weight* is (w); then the manner in v this cord will perform its vibrations is very si to that of the weight (w) attached to the m point (C) of the cord, when the cord is w weight. For let this elastic cord be brought int position AwB by some force (φ) acting in direction CE, and let the *tension of the cord* the force with which it tends to restore itself former position ACB) be equivalent to the *weigh* then the magnitude of this force (φ) at the poin will be represented by $\frac{4Px}{l}$ as before; and the force (φ) *ceases to act*, the *moving force* which the middle point of the cord approache point C will be equal to $\frac{4Px}{l}$. With respect t *mass moved*, although its *weight* is in each case *same* (viz. the *weight w*), its *inertia* will be *less* in present case than in the former; for in the fo case *each* particle of the weight (w) moves over *same* space in the time (t), but in the latter (w the weight w is spread through the *whole lengt* the cord) the several particles of which i comp

composed move *in that time* over spaces continually decreasing from E towards A or B, and consequently with a *velocity* continually diminished. Suppose therefore that the *inertia* of (w) in this present case $= qw$ (where q is some proper fraction), then the *accelerative force* with which the middle point of the cord approaches towards C will be measured by $\frac{4Px}{qwl}$.

6. If the quantity (q) be considered as *constant* (which it may be *without any material error* for cords of different lengths and weights, provided that they be composed of the *same materials* uniformly disposed), then the time of *one vibration* of this elastic cord will be found, by a process similar to that in Art. 3 and 4, to be equal to $\sqrt{\frac{qwl}{8mP}} \times \pi$ *seconds;* and the *number of vibrations in* 1″ will be equal to $\sqrt{\frac{8mP}{qwl}} \times \frac{1}{\pi}$. Hence, since q, m and π are constant quantities, the *times of vibration* of elastic cords of different lengths (l) and weights (w), when stretched by different forces (P), will *vary* as $\sqrt{\frac{wl}{P}}$; and *the number of vibrations in* 1″ will vary as $\sqrt{\frac{P}{wl}}$.

7. Let

7. Let d = the *diameter* of a cylindrical cord; then, supposing it to be of uniform density, its *weight* (w) will be measured by $\pi l d^2$, which *varies* as ld^2; in cylindrical cords, therefore, composed of the same materials, the *times* of vibration will vary as $\sqrt{\frac{l^2 d^2}{P}}$ or as $\frac{ld}{\sqrt{P}}$; and the *number of vibrations in* 1″ will vary as $\frac{\sqrt{P}}{ld}$. Hence, when the cords are stretched by the *same forces*, the *times* of vibration will be as their lengths and diameters conjointly; in cords of the *same length*, these times will be as their *diameters*; and in cords of the *same diameter*, as their *lengths*. If the cords be stretched by *different forces*, then, in the *same* cord, the times of vibration will be in an inverse sub-duplicate ratio of the forces; in cords of the *same length*, they will be in a direct ratio of the diameters and an inverse sub-duplicate ratio of the forces; and in cords of the *same diameter*, they will be directly as their lengths and in an inverse sub-duplicate ratio of the forces. The *number of vibrations in* 1″ are as these respective ratios taken *inversely*.

For

For the further illustration of this subject, the Reader is referred to the 4th Section of *Mr. Atwood's Treatise on the Rectilinear Motion and Rotation of Bodies,* and to the Authors mentioned in the margin of that work.

Lecture XXI.

ON THE MOTION OF BODIES IN CYCLOIDA AND CIRCULAR ARCS.

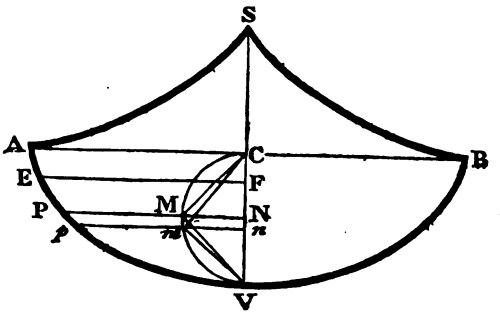
In the XIXth Lecture we compared the time a vibration in a cycloid with the time of a bod falling down the *axis* of the cycloid. We n proceed to find the actual times of vibration only in *cycloidal* but also in *circular* arcs, from principles laid down in the last Lecture.

LXX.

On the times of vibration in Cycloidal Ar

1. Let AVB be a cycloid, in which a body (suspended from the point S, performs its vibrati between the two *semi-cycloids* SA, SB, in manner described in Art. 4, page 156. Le begin its vibrations from the point E, and let be any very small arc described in the course of descent from E to the lowest point V; dr EF, PN, pn parallel to the base (AB) of the cycl and join CM, Cm, MV, mV. Let CV (the c me

meter of the generating circle) $= d$, $FV = a$, $NV = x$; then $FN = a - x$, $MV = \sqrt{CV \times NV}$



$= \sqrt{dx}$. Now by Prop^{y} III, page 154, the *cycloidal arc* PV = twice the *chord* $MV = 2\sqrt{dx}$; let $EP = z$, then $EV = z + PV$, and since EV is a *constant* quantity $\dot{z} + \dot{PV} = 0$, $\therefore \dot{z} = -$*the fluxion of* $PV = -\frac{\sqrt{d}.\dot{x}}{\sqrt{x}}$.

The *velocity* down EP (v) = the velocity down FN $= \sqrt{4m \times FN} = \sqrt{4m.\overline{a-x}}$; if therefore t = the *time* down EP, we shall have

$$i = \frac{\dot{z}}{v} = -\frac{\sqrt{d}.\dot{x}}{\sqrt{x}} \times \frac{1}{\sqrt{4m.a-x}} = -\sqrt{\frac{d}{4m}} \times \frac{x}{\sqrt{ax-x^2}};$$

and $t = -\sqrt{\frac{d}{4m}} \times$ cir. arc, rad. $= 1$, ver. sin. $\frac{x}{\frac{1}{2}a} +$ Cor^{n};

But when $t = 0$, $x = a$,

$\therefore 0 = -\sqrt{\frac{d}{4m}} \times$ circ. arc. rad. $= 1$, ver. sin. $= 2$, + Cor^{n}.

$= -\sqrt{\frac{d}{4m}} \times$ semicircle, rad. $= 1$, + Cor^{n}.

$= ($if $\pi = 3.1416$, &c.$) -\sqrt{\frac{d}{4m}} \times \pi + C$, or $C = \sqrt{\frac{d}{4m}} \times \pi$.

The

The fluent properly corrected therefore, is

$t = \sqrt{\frac{d}{4m}} \times \pi$ — cir. arc, *radius* 1, *versed sine* $\frac{x}{\frac{1}{2}a}$

$= (\text{when } x=0)\ \sqrt{\frac{d}{4m}} \times \pi$

= the time of descent through the cycloidal arc E

consequently

the *whole time of one vibration* $= \sqrt{\frac{d}{4m}} \times 2\pi = \sqrt{\frac{d}{m}} \times$

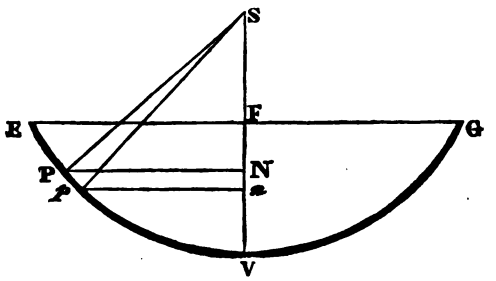
2. Since the quantity (a) does not enter into this value of (t), it is evident that the times of vibration will be the *same* from whatever point of the cycloid the body (P) begins to vibrate. Moreover, since the time down the axis $CV = \sqrt{\frac{d}{m}}$, we have the *time of a vibration* : the *time down the axis* :: $\sqrt{\frac{d}{m}}$ $\times \pi : \sqrt{\frac{d}{m}} :: \pi : 1$:: the *circumference* of a circle : its *diameter*, which coincides with what was proved in Art. 5, page 158.

LXXI.

On the method of finding the actual times of Vibration in Circular Arcs.

1. Let the small body (P) vibrating in the circular arc EVG, begin its oscillations from the point E; let $PV = z$, $NV = x$, $FV = a$, $Pp = \dot{z}$, SP

$SV=r$, then $\dot{z}=\frac{r\dot{x}}{\sqrt{2rx-x^2}}$. Let t=the time through EP, v=the velocity in the point P, then



$$v=\sqrt{4m\times FN}=\sqrt{4m.a-x}, \text{ and}$$

$$\dot{t}=\frac{-\dot{z}}{v}=\frac{-r\dot{x}}{\sqrt{2rx-x^2}}\times\frac{1}{\sqrt{4m.\overline{a-x}}}$$

$$=-\frac{r}{\sqrt{4m}}\times\frac{\dot{x}}{\sqrt{ax-x^2}}\times\frac{1}{\sqrt{2r-x}}$$

(for $\sqrt{2rx-x^2}=x^{\frac{1}{2}}\sqrt{2r-x}$); hence

$$\dot{t}=-\frac{r}{\sqrt{4m}}\times\frac{\dot{x}}{\sqrt{ax-x^2}}\times\frac{1}{\sqrt{2r}}\times\overline{1+\frac{x}{4r}+\frac{3x}{32r^2}+\&c.}$$

by expanding $\frac{1}{\sqrt{2r-x}}$ into a series.

2. Let $\frac{\dot{x}}{\sqrt{ax-x^2}}=\dot{A}$;

then, since $\frac{r}{\sqrt{4m}}\times\frac{1}{\sqrt{2r}}=\sqrt{\frac{r}{8m}}$, we have

$$\dot{t}=-\sqrt{\frac{r}{8m}}\times\overline{\dot{A}+\frac{x\dot{A}}{4r}+\frac{3x^2\dot{A}}{32r^2}+\&c.}$$

$$\therefore t=-\sqrt{\frac{r}{8m}}\times A+\frac{aA-2P}{8r}+\frac{9a^2A-18aP-12Q}{256r^2}\,^{(a)}+\&c.+Co$$

(where $P=\sqrt{ax-x^2}$, $Q=x\sqrt{ax-x^2}$, &c. &c.)

Now when $t=0$, $x=a$, $\therefore P=0$, $Q=0$, &c.; an
at the same time,

$$A\left(=f:\frac{\dot{x}}{\sqrt{ax-x^2}}\right.=\text{circ. arc, whose } \textit{rad.} \text{ is 1, } \textit{ver. sin.}\frac{x}{\frac{1}{2}a}$$

= a *semicircle* whose radius is $1=\pi$;

hence

$$0=-\sqrt{\frac{r}{8m}}\times\pi+\frac{a\pi}{8r}+\frac{9a^2\pi}{256r^2}+\&c.+\text{Cor}^n.$$

Th

(a) Assume $\sqrt{ax-x^2}=P$,

then $\frac{\frac{1}{2}a\dot{x}-x\dot{x}}{\sqrt{ax-x^2}}=\dot{P}$,

and $\frac{x\dot{x}}{\sqrt{ax-x^2}}=\frac{a\dot{A}-2\dot{P}}{2}=x\dot{A}$,

$$\therefore f:\frac{x\dot{A}}{4r}=f:\frac{a\dot{A}-2\dot{P}}{8r}=\frac{aA-2P}{8r}.$$

Again, let $x\sqrt{ax-x^2}=Q$,

then $\frac{\frac{3}{2}ax\dot{x}-2x^2\dot{x}}{\sqrt{ax-x^2}}=\dot{Q}$,

and $\frac{x^2\dot{x}}{\sqrt{ax-x^2}}=\frac{3ax\dot{A}-2\dot{Q}}{4}=\frac{3a^2\dot{A}-6a\dot{P}-4\dot{Q}}{8}$,

$$\therefore f:\frac{3x^2\dot{A}}{32r^2}=f:\frac{9a^2\dot{A}-18a\dot{P}-12\dot{Q}}{256r^2}=\frac{9a^2A-18aP-1}{256r^2}$$

In the same manner, the fluent of the nth term of the ser
might be found in functions of the preceding ones, by assumi
$x^{n-1}\times\sqrt{ax-x^2}=R$, &c.; but when the arcs in which the pe
dulum vibrates are small, this series converges so fast, that *th*
terms are amply sufficient for all ordinary purposes.

The fluent, therefore, when properly corrected, is $t=\sqrt{\frac{r}{8m}}\times\pi\times\overline{1+\frac{a}{8r}+\frac{9a^2}{256r^2}+\&c.}-\sqrt{\frac{r}{8m}}\times\overline{A+\frac{aA-2P}{8r}+\&c.}$ $=\sqrt{\frac{r}{8m}}\times\pi\times\overline{1+\frac{a}{8r}+\frac{9a^2}{256r^2}+\&c.}$ (when $x=0$, and consequently A, P, Q, &c. each=0) for the time of descent down the arc EV; $\therefore$ *the whole time of one vibration*= $\sqrt{\frac{r}{8m}}\times 2\pi\times\overline{1+\frac{a}{8r}+\frac{9a^2}{256r^2}+\&c.}$

3. If the arcs be so very small that the *versed sine* (a) may be entirely rejected with respect to the *radius* (r), then the time of a *vibration*$=\sqrt{\frac{r}{8m}}\times 2\pi=$ (if $r=2d$)$\sqrt{\frac{d}{4m}}\times 2\pi=\sqrt{\frac{d}{m}}\times\pi=$ the time of a vibration in a *cycloid* whose axis is d or $\frac{1}{2}r$ (Art. 1, p. 200). The time of oscillation therefore in a very small circular arc is the same as the time of oscillation in a cycloid whose axis is equal to $\frac{1}{2}$ the length of the string by which the body is suspended; this confirms what was said in Art. 5, page 158[b].

4. If

(b) In these very small arcs, the *time of descent down* EV $=\sqrt{\frac{r}{8m}}\times\pi$. Now the time down the *chord*=the time down the *diameter* of a circle $=\sqrt{\frac{2r}{m}}$. Hence the time down a *very small circular arc* : the time down the *corresponding cord* $::\sqrt{\frac{r}{8m}}\times\pi:\sqrt{\frac{2r}{m}}::\pi\sqrt{\frac{1}{8}}:\sqrt{2}::\pi:4::$ the *circumference* of a circle : four of its *diameters*.

4. If the length of the arc through which the pendulum vibrates be such that all the terms of the series except the *two first* may be rejected, then the time of a vibration $= \sqrt{\frac{r}{8m}} \times 2\pi \times \overline{1+\frac{a}{8r}}$; and consequently, the time of a vibration in a *circular arc* whose radius is (r) : the time of a vibration in a *cycloid* whose axis $= \frac{1}{2}r :: 1 + \frac{a}{8r} : 1$. Let the arc through which the pendulum vibrates on each side the lowest point be 2°, then $a =$ *versed sine of* 2° $= \frac{6r}{10000}$ nearly; $\therefore \frac{a}{8r} = \frac{1}{13333}$. In arcs which are less than 2°, this difference will be still less; so that if the whole arc through which a pendulum vibrates does not *exceed* 4°, it may without any material error be considered as oscillating in a *cycloid*, and consequently all its oscillations will be performed in the *same time*, whether it vibrate in a greater or a lesser arc *within this limit*.

5. Let $A =$ the arc whose *versed sine* is (a), then (supposing the arc to be *very small*) $a = \frac{A^2}{2r}$ [a]; and if $n =$ the number of degrees in this arc, then A:

(a) For the *versed sine* of an arc $= \frac{\overline{\text{chord}}^2}{\text{diam.}}$; but if the arc be *very small*, then the *chord* is equal to the *arc*, $\therefore a = \frac{\overline{\text{arc}}^2}{\text{diam.}} = \frac{A^2}{2r}$.

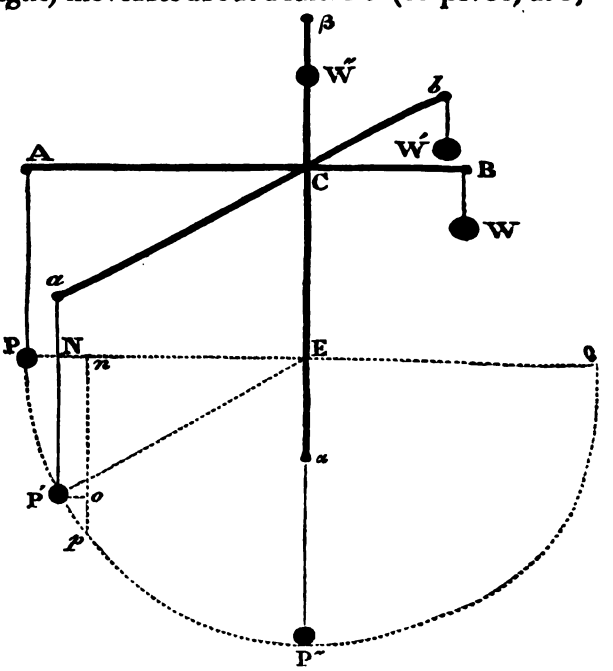
$A : 2\pi r :: n : 360$, or $A = \frac{2\pi r n}{360}$, $\therefore a = \frac{A^2}{2r} = \frac{2\pi^2 r n^2}{360^2}$ $= (\text{for } \pi = 3.1416 \text{ \&c.}) \frac{rn^2}{6565}$ very nearly; hence $\frac{a}{8r} = \frac{n^2}{52520}$ = the variation in an arc of n degrees. For the same reason, the variation in an arc of N degrees $= \frac{N^2}{52520}$; consequently the variation for $N-n$ degrees is $\frac{N^2-n^2}{52520}$; if therefore the pendulum keeps *true time* when it oscillates through n degrees, the *error* arising from its vibrating through N degrees will be $\frac{N^2-n^2}{52520}$ *of the time of one vibration.* For instance, if a pendulum which oscillates *seconds* keeps *true time* when it vibrates through an arc of 2° on each side the vertical line, then if it vibrates through an arc of 3°, it will *lose* $\frac{3^2-2^2}{52520} = \frac{5}{52520}$ $= \frac{1}{10504}$ *every second,* or $\frac{60 \times 60 \times 24}{10504} = 8.2 =$ about 8″ in a *day.* If it be made to vibrate through an arc of 2°. 10′ on each side the vertical, then it will lose only $\frac{\overline{2.16 \text{ \&c.}}^2 - 2^2}{52520} = \frac{.665 \text{ \&c.}}{52520}$ in a *second,* i. e. $\frac{60 \times 60 \times 24 \times .665 \text{ \&c.}}{52520}$ (or about one second) in a *day.*

LXXII.

LXXII.

On the angular Motion of two bodies suspend from the extremities of the arms of straight Lever.

1. Let ACB be a straight lever (of inconsidera weight) moveable about a fulcrum (or pivot) at C, a

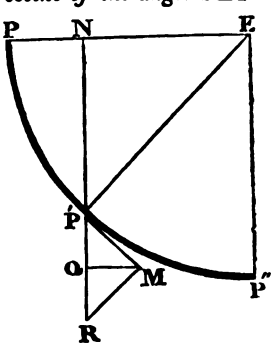


from the extremities of its arms CA, CB, let weights P, W be suspended. Draw $\beta CB''$ at ri angles to ACB, and complete the parallelogr $APEC$; then, whilst the lever moves from t *horizontal* position BCA to the *vertical* one $\beta($ the weight P will describe the circular arc PP'.

wh

whose *radius* is EP or CA, and *center* E; see Exam. 8, page 247, Vol. I. Let $AC=r$, $BC=\varrho$; and when the lever is in the position aCB, let $P'N$ (the sine of the angle PEP') $=x$. In the *horizontal* position (ACB) of the lever, the weight which would keep the weight W in equilibrio is $\frac{W\varrho}{r}$; $\therefore$ the *moving force* with which P has a tendency to descend from this position is $P-\frac{W\varrho}{r}=\frac{Pr-W\varrho}{r}$; the *inertia* of W as opposed to $P=\frac{W\varrho^2}{r}$, $\therefore$ the *mass moved* $=P+\frac{W\varrho^2}{r^2}=\frac{Pr^2+W\varrho^2}{r^2}$; hence the *accelerative force* with which P *begins* to descend $=\frac{Pr-W\varrho}{r}\times\frac{r^2}{Pr^2+W\varrho^2}=\frac{Pr^2-Wr\varrho}{Pr^2+W\varrho^2}=\varphi$. Now the accelerative force in the circular arc at P (φ) : the force at P' (F) :: EP' : EN[a] :: r : $\sqrt{r^2-x^2}$, therefore

([a]) The accelerative force with which a heavy body describes the quadrantal arc $PP'P''$ varies as the *cosine of the angle* PEP'. For let $P'R$ represent the gravity of the body, and resolve it into PM' *perpendicular* and MR *parallel* to EP'; then $P'M$ is that part of gravity which accelerates the motion of the body in the curve; $\therefore$ the gravity of the body (1) : *accelerative force* in the curve (F) :: $P'R$: PM' :: (by sim. Δ^s) EP' : EN, $\therefore F=\frac{1\times EN}{EP'}$, which *varies* as EN, EP' being *constant*. For this reason, the accelerative force (φ), with which the weight (P) *begins* to descend from the horizontal position EP, will be diminished at any point (P') in the ratio of EN : EP'.



$\therefore F=\frac{\varphi\sqrt{r^2-x^2}}{r}$; and this is the variable force with which the weight (P) describes the circular arc $PP'P''$.

2. Let Pp be any very small part of the arc: draw pn parallel to PN, and Po parallel to PE: let $PP'=z$, then $Pp=\dot{z}=\frac{r\dot{x}}{\sqrt{r^2-x^2}}$. Let v = the velocity generated in the point P' by the force (F) acting along the arc $PP'p$, then

$v\dot{v}=2mF\dot{z}=$(since $F=\frac{\varphi\sqrt{r^2-x^2}}{r}$, and $\dot{z}=\frac{r\dot{x}}{\sqrt{r^2-x^2}}$)

$$2m\times\frac{\varphi\sqrt{r^2-x^2}}{r}\times\frac{r\dot{x}}{\sqrt{r^2-x^2}}$$

$$=2m\varphi\dot{x};$$

$\therefore v^2=4m\varphi x$, and $v=2\sqrt{m\varphi x}=$ (when $x=r$) $2\sqrt{m\varphi r}$. Hence (by Art. 2, page 3) the *velocity at any point* P' is equal to the velocity which a body would acquire by falling *perpendicularly* through the space (x) by the action of the constant force (φ); and when the lever has descended from its *horizontal* position BC to the *vertical* one $\beta C\alpha$, the velocity of the weight P (in an horizontal direction) is equal to the velocity which a body would acquire by falling perpendicularly through the length of the longer arm CA by the action of the force (φ).(a)

3. With

(a) With the velocity thus acquired, the weight P would have a *tendency* to ascend through the arc $P''Q$ equal to PP'' *on the other side of the lowest point* P'', and so go on to perform its oscillations

3. With respect to *the pressure upon the fulcrum or pivot*, it is evident that when the lever is in the *horizontal* position ACB, this pressure is the same as for the *wheel and axle* whose radii are respectively r and ϱ, which (when $w=0$, in Note b, pp. 83, 84) is $P+W-\frac{\overline{Pr-W\varrho}^2}{Pr^2+W\varrho^2}$. To find how this pressure *varies* during the descent of the lever from an horizontal to a vertical position, let the force with which the common center of gravity of P and W $\left(\text{viz. } \frac{\overline{Pr-W\varrho}^2}{Pr^2+W\varrho^2}\right)$ *begins* to descend, be represented by $P'R$ in Figure of Note (a), page 207; let $P'R$ be resolved into $P'M, MR$, and then let $P'M$ be resolved into $P'Q, QM$; PQ will represent that part of the force $\left(\frac{\overline{Pr-W\varrho}^2}{Pr^2+W\varrho^2}\right)$ of which the fulcrum is relieved when the lever is in the position EP'. But $P'Q : P'R \left(\frac{\overline{Pr-W\varrho}^2}{Pr^2+W\varrho^2}\right) :: PM^2 : PR^2 ::$ (by sim Δ^s) $EN^2 : EP'^2 ::$ cos. $\overline{PEP'}^2 :$ $\overline{\text{rad.}}^2$ (1), $\therefore PQ' = \frac{\overline{Pr-W\varrho}^2}{Pr^2+W\varrho^2} \times \overline{\cos. PEP'}^2$. Hence the pressure on the fulcrum, when the lever is in the position

oscillations in the semicircle $PP''Q$; but it is evident that this tendency would be so counteracted by the friction on the axis, the entanglement of the string to which the weight (W) is attached, &c. &c. that the motion of the lever would soon cease.

pressure $= P - W - \frac{\overline{P_1 - W_1}^2}{P_1^2 - W_1^2} \times \cos. PEP_1$;

and when the lever is arrived at the *vertical* position PE, $\cos. PEP_1 = 1$, $\therefore$ the pressure upon it $= P - W$ as it ought to do).

4. Let r = the time in which P moves through the arc PP_1, then

$$= \frac{2}{1} \cdot \frac{\sqrt{2}}{\sqrt{1-z^2}} \cdot \frac{1}{1 \cdot \sqrt{2}z} \cdot \frac{r}{\sqrt{2}} \times \frac{z}{1 \cdot z^2} \times \frac{1}{\sqrt{1-z^2}}$$

$$= \frac{r}{\sqrt{2}} \cdot \frac{z}{1 \cdot z^2} \cdot \frac{1}{1} \cdot \frac{z^2}{2} \cdot \frac{1 \cdot 3 \cdot z^4}{2 \cdot 4} \cdot \frac{1 \cdot 3 \cdot 5 \cdot z^6}{2 \cdot 4 \cdot 6} + \&c.$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{z}{1 \cdot z^2} \cdot \frac{z}{1} \cdot \frac{z^2 z}{2} \cdot \frac{1 \cdot 3 \cdot z^4 z}{2 \cdot 4} \cdot \frac{1 \cdot 3 \cdot 5 \cdot z^6 z}{2 \cdot 4 \cdot 6} + \&c. \quad \text{therefore}$$

$$= \sqrt{\frac{2}{r}} \cdot z \left(1 + \frac{z^2}{2 \cdot 3} + \frac{1 \cdot 3 \cdot z^4}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 \cdot z^6}{2 \cdot 4 \cdot 6 \cdot 7} + \&c.\right)$$

$$= \text{the time} = \sqrt{\frac{2}{r}} \cdot z \left(1 + \frac{1}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} + \&c.\right) = 1.2025 \sqrt{\frac{2}{r}} \cdot z$$

When $r = 1$, $z = 0$, this fluent therefore needs a *correction*: and though the series converges but very slowly when $z = 1$, yet the expression $1.2025 \sqrt{\frac{2}{r}} \cdot z$ for the time in which the lever *descends from the horizontal to the vertical position* is a very near approximation to its true value[a]. If the lever descends only through small angles, then $\frac{z}{r}$ is a *fraction*, and the series

[a] The number 1.2025 is the sum of the first seven terms of the series, and is true in the third place of decimals.

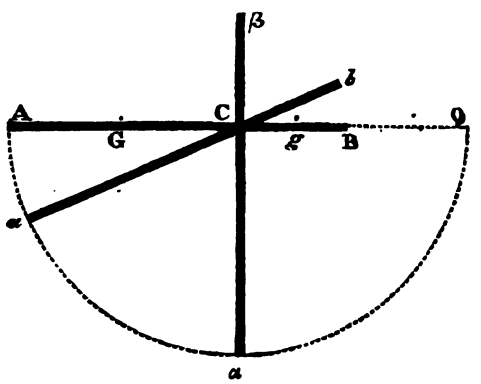
series converges very fast; for instance, let the angle PEP' or ACa be 30°, then $x=\frac{1}{2}r$, and (neglecting all the terms of the series after the *third*) we have

$$t=\sqrt{\frac{r}{2m\varphi}}.\overline{1+\frac{1}{40}+\frac{1}{384}}$$

$$=1.0276\sqrt{\frac{r}{m\varphi}} \text{ seconds.}$$

If $P=5$ lbs, $W=6$ lbs. $r=5$ feet, $\varrho=4$ feet, then $\varphi=\frac{Pr^2-Wr\varrho}{Pr^2+W\varrho^2}=\frac{5}{221}$, and $t=1.0276\sqrt{\frac{5\times 221}{32\frac{1}{6}\times 5}}$ $=2.6923$ seconds.

5. The foregoing principles may be applied to finding the time in which a rod or small *cylinder*



ACB of uniform density (moveable round a pivot passing through C) descends from an horizontal to a vertical position. For let $CA=r$, $CB=\varrho$; then, supposing the weights of the parts CA, CB to be collected

collected into their respective *centers of gravity* G, g, the *moving force* with which the point G has a tendency to descend is $r-\frac{\varrho^2}{r}$ [a] $=\frac{r^2-\varrho^2}{r}$, and the *mass moved* is $\frac{4.\overline{r^3+\varrho^3}}{3r^2}$ [b]; $\therefore$ the *accelerative force* of the point G is $\frac{r^2-\varrho^2}{r}\times\frac{3r^2}{4.r^3+\varrho^3}=\frac{3r.\overline{r^2-\varrho^2}}{4.r^3+\varrho^3}$; and since $CA : CG :: 2 : 1$, the force which accelerates *the point* $A=\frac{3r.\overline{r^2-\varrho^2}}{2.r^3+\varrho^3}$ (see Art. 11, page 69). Hence, in the descent of the extremity (A) of the cylinder through the circular arc $Aa\alpha$, φ is equal to $\frac{3r.\overline{r^2-\varrho^2}}{2.r^3+\varrho^3}$; $\therefore$ (by Art. 2) the *velocity* which the point

(a) For since the cylinder is of uniform density, the *weights* of the parts CA, CB, may be represented by their respective *lengths*; hence the *weight placed in* $G=r$; and the weight which placed in G would balance the *weight* ϱ placed in g (w) : the weight (ϱ) $::$ $Cg : CG :: CB$ (ϱ) : CA (r), $\therefore w=\frac{\varrho^2}{r}$; consequently the *moving force* of the weight collected in $G=r-w=r-\frac{\varrho^2}{r}$.

(b) Let x=the distance of any particle of the rod from the center of motion C, then the *moment of inertia* for the part $CA=\int x^2\dot{x}$ when $x=r$ ($=\frac{1}{3}r^3$); and the moment of inertia for the part $CB=\int x^2\dot{x}$ when $x=\varrho$ ($=\frac{1}{3}\varrho^3$). The *inertia of the rod* therefore, as referred to the point $G=\frac{\frac{1}{3}r^3+\frac{1}{3}\varrho^3}{CG^2}=\frac{\frac{1}{3}r^3+\frac{1}{3}\varrho^3}{\frac{1}{4}r^2}=\frac{4.\overline{r^3+\varrho^3}}{3r^2}$.

point A will have acquired when the cylinder is come to the vertical position $\beta C\alpha = (2\sqrt{m\varphi r} =)$ $2\sqrt{\frac{3mr^2.\overline{r^2-\varrho^2}}{2.r^3+\varrho^3}}$, and (by Art. 4, p. 210) the *time* of its descent $= \left(1.2025\sqrt{\frac{r}{m\varphi}} = \right) 1.2025\sqrt{\frac{2.r^3+\varrho^3}{3m.\overline{r^2-\varrho^2}}}$.

6. Let $r : \varrho :: 2 : 1$, or $r = 2\varrho$ (in which case A will be the *center of oscillation* of the line or small cylinder), then $\varphi = \frac{3r.\overline{r^2-\varrho^2}}{2.r^3+\varrho^3} = 1$; $\therefore$ the velocity of the point A, when the cylinder comes to the vertical position $\beta C\alpha$ $(= 2\sqrt{m\varphi r}) = 2\sqrt{mr} =$ *the velocity which a heavy body would acquire in falling freely by the force of gravity through the space* (r). The *time* of descent $= 1.2025\sqrt{\frac{r}{m}} = 1.2025 \times$ *the time of a body's falling freely through the space* (r); and consequently the time of an *oscillation* through the semicircle $A\alpha Q = 2.405 \times \sqrt{\frac{r}{m}} = 2.405 \times$ time of falling through (r).

Lecture XXII.

ON THE RECTILINEAR MOTION OF BODIES BY THE VARIABLE FORCE OF GRAVITY.

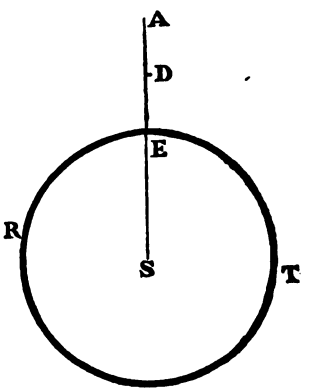
In the second Lecture we found the relation between the *space*, *time* and *velocity* of a body ascending or descending near the Earth's surface, on supposition that the force of gravity was a *constant* force. But it has already been observed (Note a, page 162) that this force is not constant, but varies *inversely as the square of the distance from the Earth's center*; we shall conclude therefore with shewing the method of ascertaining the actual time and velocity of a body descending from any point above the Earth's surface, on supposition that the force of gravity varies according to this law.

LXXIII.

On the method of ascertaining the actual velocity acquired by a body descending perpendicularly towards the Earth's surface, on supposition

supposition that the force of gravity varies inversely as the square of the distance from the Earth's center.

1. Let ERT represent the Earth, S its center, A the point from which the body begins to descend; and let $SA=a$, $SE=r$, $SD=x$, $m=16\frac{1}{12}$ feet, $v=$ the *velocity* which the body has acquired in falling through AD, and the force of gravity at the Earth's surface $=1$. Since the force of gravity varies *inversely as the square of the distance from* S, the force *at* E (1) : the force *at* D (F) :: $SD^2 : SE^2 :: x^2 : r^2$, $\therefore F=\frac{r^2}{x^2}$; but (since the velocity *increases* as x *decreases*) $v\dot{v}=-2mF\dot{x}$; hence



$$v\dot{v}=\frac{-2mr^2\dot{x}}{x^2}, \text{ and } \frac{v^2}{2}=\frac{2mr^2}{x}+\text{Cor}^n;$$

now when $v=0$, $x=a$, $\therefore 0=\frac{2mr^2}{a}+C$,

consequently $\frac{v^2}{2}=\frac{2mr^2}{x}-\frac{2mr^2}{a}$,

$$\text{or } v^2=\frac{4mr^2.\overline{a-x}}{ax}$$

$$=(\text{when } x=r)\ \frac{4mr.\overline{a-r}}{a}$$

equal

$= 4mr \times \frac{AE}{SA}$; the whole velocity therefore which a body would acquire in falling *from the point A to the Earth's surface* $= 2\sqrt{mr} \times \sqrt{\frac{AE}{SA}}$ feet in 1″.

2. Since $r = 3970$ *miles* $= 3970 \times 5280$ *feet*, $2\sqrt{mr}$ $= 2\sqrt{16\frac{1}{12} \times 3970 \times 5280} = 36722$ feet; hence the velocity acquired in falling through any space AE may be generally expressed by $36722\sqrt{\frac{AE}{SA}}$ feet in 1″. Thus let $AE = 1, 2, 3, 4$ &c. miles, then $SA = 3971$, 3972, 3973 &c. miles; and the velocity acquired in falling through 1, 2, 3 &c. miles above the Earth's surface is equal to $36722\sqrt{\frac{1}{3971}}$, $36722\sqrt{\frac{2}{3972}}$, $36722\sqrt{\frac{3}{3973}}$, &c. or to 582, 834, 1020, &c. feet in 1″ respectively.

3. The velocity acquired in falling through AE by the force of gravity at the Earth's surface continued *constant* $= 2\sqrt{m \times AE}$; hence the velocity acquired by the *constant* force (V) : the velocity acquired by the *variable* force (v) :: $2\sqrt{m \times AE}$: $2\sqrt{mr} \times \sqrt{\frac{AE}{SA}}$:: $\sqrt{SA}$: $\sqrt{r}$:: $\sqrt{SE + AE}$: $\sqrt{SE}$:: (if AE be *very small* with respect to SE) $SE + \frac{1}{2}AE$[a] : SE; $\therefore V - v : v :: \frac{1}{2}AE : SE$, or

$V - v$

(a) For when two quantities are very nearly equal, the ratio of their *square roots* is found by *halving their difference*. (See Algebra, Lect. 8.)

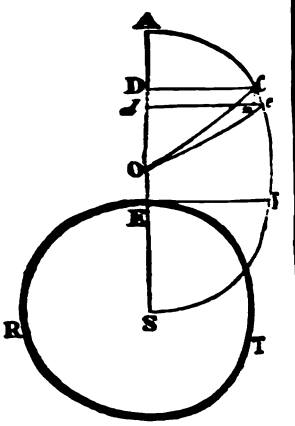
$V-v=\frac{AE}{2SE}\times v$. Now since V is the velocity acquired by the body on supposition that the force is *constant*, and v is the velocity *actually* acquired by it, $V-v$ will be the *error* arising from that supposition; which error will be found to be very inconsiderable for small distances above the Earth's surface. For instance, let $AE=1$ mile, then $V-v=\frac{AE}{2SE}\times v=$ (for $SE=3970$) $\frac{v}{7940}$; and since (by Art. 2.) $v=582$ feet, the error arising from the supposition of the constant force is only $\frac{582}{7940}$, or about $\frac{1}{14}$th of a foot. For all distances therefore *within a mile of the Earth's surface*, the common method of calculating the velocity acquired by a body descending perpendicularly is true to a great degree of exactness.

LXXIV.

On the Method of calculating the Time *in which a body descends in the foregoing manner.*

1. Suppose the body to begin to descend from the point A as before (Fig. in p. 218.), and upon SA describe the semicircle AFS; from D draw DC at right angles to SA; let the *arc* $AC=A$, and its *sine*

$CD=S$, then (retaining the notation of the preceding Section) $\dot{A}=\frac{-\frac{1}{2}a\dot{x}}{\sqrt{ax-x^2}}$,[a] $S=\sqrt{ax-x^2}$, and $\dot{S}=\frac{\frac{1}{2}a\dot{x}-x\dot{x}}{\sqrt{ax-x^2}}$; from which



$$\frac{-x\dot{x}}{\sqrt{ax-x}}=-\frac{\frac{1}{2}a\dot{x}}{\sqrt{ax-x^2}}+\dot{S}$$
$$=\dot{A}+\dot{S}.$$

Now, by Art. 1, page 215,

$$v^2=\frac{4mr^2.\overline{a-x}}{ax},$$
$$\therefore v=\frac{2r\sqrt{m}\times\sqrt{a-x}}{\sqrt{ax}};$$

but, if $t=$ the *time* of falling through AD, then

$$\dot{t}=\frac{-\dot{x}}{v}=\frac{\sqrt{a}}{2r\sqrt{m}}\times\frac{-x^{-\frac{1}{2}}\dot{x}}{\sqrt{a-x}}=\frac{\sqrt{a}}{2r\sqrt{m}}\times\frac{-x\dot{x}}{\sqrt{ax-x^2}}$$
$$=\frac{\sqrt{a}}{2r\sqrt{m}}\times\overline{\dot{A}+\dot{S}};\text{ hence}$$
$$t=\frac{\sqrt{a}}{2r\sqrt{m}}\times\overline{A+S}$$
$$=\frac{\sqrt{a}}{2r\sqrt{m}}\times\overline{AC+CD},$$

which

(a) For draw cd indefinitely near and parallel to CD, and (O being the *center* of the semicircle) join CO, cO; draw Cn parallel to Dd; then Cc $(\dot{A})$: Cn or Dd $(-\dot{x})$:: CO $(\frac{1}{2}a)$: CD $(\sqrt{x.a-x})$, $\therefore \dot{A}=\frac{-\frac{1}{2}a\dot{x}}{\sqrt{ax-x^2}}$.

which needs no *correction*, for when $t=0$, $A+S=O$. If EF be drawn at right angles to SA, then *the whole time of descending through* $AE=\frac{\sqrt{a}}{2r\sqrt{m}}\times\overline{AF+FE}$ $=\frac{1}{2r\sqrt{m}}\times\sqrt{SA}\times\overline{AF+FE}$ seconds.

2. Since $m:r::16\frac{1}{12}:3790\times5280$, let $r=1$, then $m=\frac{193}{251539200}=\frac{1}{1303311}$, and $\sqrt{m}=\frac{1}{1141}$; hence $\frac{1}{2r\sqrt{m}}=\frac{1}{2\sqrt{m}}=570.81$. If therefore SA, AF, FE be expressed in terms of the *Earth's radius*, $570.5\times\sqrt{SA}\times\overline{AF+FE}$ *seconds* will be a general expression for the time of falling through any space AC above the Earth's surface; but as it involves the *length of an arc* (AF) whose versed sine is AE, it cannot be readily applied, except in particular cases.

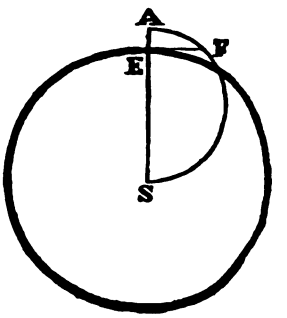
EXAM. Suppose a body to fall from an height equal to the *Earth's radius*, then $AF=$ a *quadrant* whose radius is (r); if therefore $\pi=3.14159$ and $r=1$, we shall have $t=$(for $SA=2r=2$) $570.81\times\sqrt{2}\times\overline{\frac{1}{2}\pi+1}=570.81\times1.4142\times2.5707=2075$ seconds $=34'.35''$.

3. If the space AE through which the body falls be so *small* as to bear no sensible proportion to the radius of the Earth, then the *arc* AF may be considered as equal to the *sine* FE. In this case (Fig. in p. 220) $AF+FE=2FE=2\sqrt{AE\times ES}=2\sqrt{r\times AE}$; therefore

$t=$

$$t=\left(\frac{1}{2r\sqrt{m}}\times\sqrt{SA}\times\overline{AF+FE}=\right)\frac{\sqrt{r\times SA\times AE}}{r\sqrt{m}}$$
$$=\sqrt{\frac{SA}{r}}\times\sqrt{\frac{AE}{m}}=\sqrt{\frac{SA}{SE}}\times\sqrt{\frac{AE}{m}}.$$

Now the time of falling through AE by the force of gravity considered as *constant* $=\sqrt{\frac{AE}{m}}$; hence the time by the *variable* force (t) : the time by the *constant* force (T)

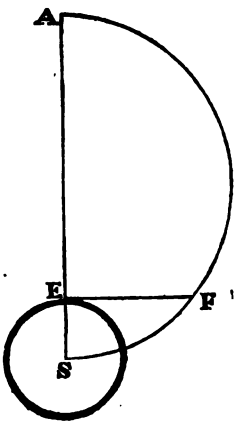


$$::\sqrt{\frac{SA}{SE}}\times\sqrt{\frac{AE}{m}}:\sqrt{\frac{AE}{m}}$$
$$::\sqrt{SA}:\sqrt{SE}$$
$$::\sqrt{SE+AE}:\sqrt{SE}::SE+\tfrac{1}{2}AE:SE;$$

consequently $t-T:T::\frac{1}{2}AE:SE$, $\therefore\ t-T=\frac{T\times AE}{2SE}$; i. e. "the *error* in time, arising from the "supposition that the force of gravity is *constant*, is "that part of the *whole time of descent* which is ex- "pressed by the fraction $\frac{AE''}{2SE}$;" this error therefore will be extremely small even for very considerable *distances* above the Earth's surface. For example, let a body fall from the height of 50 *miles*, which is about $\frac{1}{80}$th part of the earth's radius, then $\frac{AE}{2SE}=\frac{1}{160}$. Now the time of falling through 50 miles, on supposition

position that the force is *constant*, is 128 seconds; hence the *error* arising from this supposition is only $\frac{128}{160}$ or $\frac{4}{5}$th of a *second*.

4. If the distance from which the body falls is so great that the *sine* EF may be considered as equal to the *arc* FS, then $AF+FE=AF+FS$ =the *semicircle* $AFS=\frac{1}{2}\pi \times SA$. Hence, by Art. 2, the time of falling through AE (t)

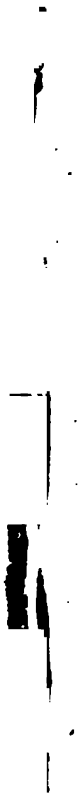


$=570.81 \times \sqrt{SA} \times \frac{1}{2}\pi \times SA$
$=570.81 \times 1.57079 \times SA^{\frac{3}{2}}$
$=896.62SA^{\frac{3}{2}}$ seconds.

This then is a general expression for the time of a body's falling to the Earth's surface, when the distance from which it falls bears a great proportion to the Earth's radius; for instance, suppose the distance of the Moon from the Earth to be 60 of those radii, then *the time of its falling to the Earth* will be equal to

$896.62 \times \overline{60}^{\frac{3}{2}}$
$=896.62 \times \sqrt{216000}=896.62 \times 464.75$
$=416704$ seconds $=4$ days 19 hours 45.′ 4″.

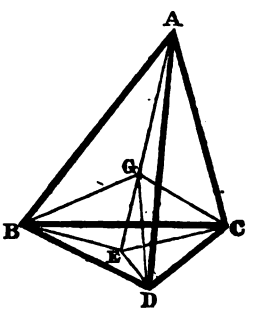
THE END.



APPENDIX.

I.

By way of *additional* corollary to Exam. II. page 103, besides that in Note *a*, page 104, it might have been observed, that "if a body be placed at the center of gravity of "a *triangular pyramid*, and if this body be acted upon by "*four forces* which are to each other as the four lines drawn "from the center of gravity to the *angular points* of the "pyramid, the body would remain *at rest*." For let *ABCD* represent the pyramid; let *E* be the center of gravity of its *base BCD*, and join *AE*; take $AG=3GE$, then *G* will be the center of gravity of the *pyramid*(a). Join *GB, GC, GB; EB, EC, ED*; then the three forces acting in directions *GB, GC, GD* may be considered as resolved into *GE, EB; GE, EC; GE, ED* respectively; of which *EB, EC, ED* (by Note *a*, page 104) destroy each other's effects; there remains therefore $3GE$, which being *equal to GA*, a force acting in direction *GA* will destroy



(a) Since the pyramid is made up of triangles similar and parallel to *BCD*, it is evident that *AE* must intersect all those triangles in a similar manner; and since it passes through the center of gravity of *one* of them, it must pass through the center of

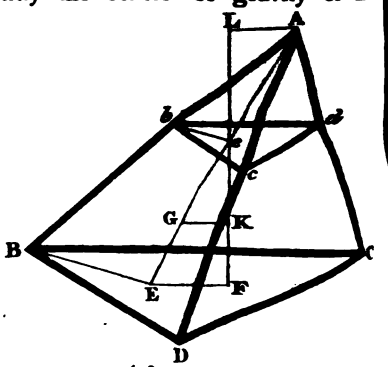
destroy its effect also, and consequently "a body acted upon "by four forces represented in quantity and direction by "*GA, GB, GC, GD*, would remain at *rest*."

II.

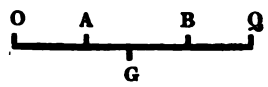
In the collision of hard and elastic bodies (Lect. V.) we omitted to shew that "the motion of their *common center* "*of gravity* is not affected by the impact of the bodies upon "each other."

1. Let *A* and *B* be two perfectly *hard* bodies moving in the direction *OQ*; suppose *A* to impinge upon *B*, and let *G* be

of gravity of them *all*; hence the *common center of gravity* of the triangles (and consequently the center of gravity of the *pyramid*) is in the line *AE*. For the purpose of finding its distance from *A*, let (*bcd*) be one of the similar triangles at the distance (*Ae*) from *A*; through (*e*) draw *LeF* at right angles, and from *A* draw *AL* parallel to the *base BCD* of the pyramid; join *EF*. Let $\Delta BCD = A$, $LF = a$, $Le = x$; then, from *similar figures*, $\Delta BCD\ (A) : \Delta bcd :: BD^2 : bc^2 :: BE^2 : be^2 :: AE^2 : Ae^2 :: LF^2\ (a^2) : Le^2,\ (x^2)$ $\therefore \Delta bcd = \frac{Ax^2}{a^2}$. Now, by Art. 1, page 129, the distance of the center of gravity of the pyramid from the line *AL* is equal to $\frac{\int bcd \times x\dot{x}}{\int bcd \times \dot{x}} = \frac{\int x^3\dot{x}}{\int x^2\dot{x}} = \frac{x^4}{4} \times \frac{3}{x^3} = \frac{3}{4}x$ = (when $x = LF$) $\frac{3}{4}LF$; take $\therefore LK = 3KF$, and draw KG parallel to *EF*, then *G* will be the center of gravity of the pyramid; and since $LK = 3KF$, *AG* will be equal to $3GE$. It is evident that this method would apply to finding the center of gravity of a pyramid whose base is *any figure whatever*, as well as to the *triangular* pyramid.



be their *common center of gravity*. Let the *variable* distance of A from the *fixt* point $O=x$, of $B=y$, and of $G=X$; then, by Exam. V. page 108, $X=\frac{Ax+By}{A+B}$. Suppose now a and b to represent the *velocities* of A and B, and that $t=$the *time* of their motion, and consequently the time of the motion of their *common center of gravity*; then (since $V=\frac{S}{T}$) the velocity of A $(a)=\frac{\dot{x}}{\dot{t}}$; of B $(b)=\frac{\dot{y}}{\dot{t}}$;



of $G=\frac{\dot{X}}{\dot{t}}$. Now $\dot{X}=\frac{A\dot{x}+B\dot{y}}{A+B}$, $\therefore \frac{\dot{X}}{\dot{t}}=\frac{\frac{A\dot{x}}{\dot{t}}+\frac{B\dot{y}}{\dot{t}}}{A+B}=\frac{Aa+Bb}{A+B}$ =the *common velocity of A and B after impact;* and as the velocity of their common center of gravity *after impact* is evidently the same with the velocity of the bodies themselves, it follows that its velocity both *before* and *after* impact is the same, i.e. "its motion is not affected by the impact of the bodies upon each other."

2. If the two bodies A and B be perfectly *elastic*, then $X=\frac{Ax+By}{A+B}$, and $\frac{\dot{X}}{\dot{t}}=\frac{Aa+Bb}{A+B}$=the velocity of their common center of gravity (G) *before* impact, as in the preceding Article. To find its velocity *after* impact, we must refer to Art. 2. page 161, where the *velocity of A after impact* $=\frac{\overline{A-B}.a+2Bb}{A+B}(\alpha)$, and of $B=\frac{\overline{B-A}.b+2Aa}{A+B}(\beta)$; now $\alpha=\frac{\dot{x}}{\dot{t}}$, and $\beta=\frac{\dot{y}}{\dot{t}}$, substituting therefore the values of $\frac{\dot{x}}{\dot{t}}$, and $\frac{\dot{y}}{\dot{t}}$ in the equation $\frac{\dot{X}}{\dot{t}}=\frac{\frac{A\dot{x}}{\dot{t}}+\frac{B\dot{y}}{\dot{t}}}{A+A}$, we have

$$\frac{\overline{A-B}.Aa+2ABb+\overline{B-A}.Bb+2ABa}{\overline{A+B}^2}=\frac{\overline{A+B}.Aa+Bb}{\overline{A+B}^2}$$

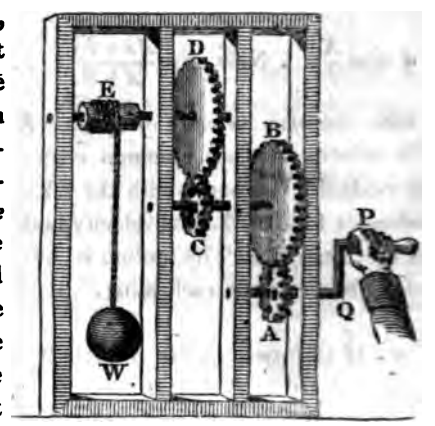
$=\frac{Aa+Bb}{A+B}$ for the velocity of the common center of gravity of

of *A* and *B after* impact; from which we draw the same conclusion as in the preceding Article.

III.

The following Note should have been inserted in Lect. VIII. page 259.

1. In a combination of wheels as represented in the annexed Figure, where the *small* wheels *A, C,* &c. act upon the *large* ones *B, D,* &c. it is evident that such parts only of the machine as are upon *the same axis of rotation* perform their revolutions in *the same time*. Suppose the teeth of the larger and smaller wheels to be *equal,* and that the *number* of teeth in the large wheel is to that in the smaller as $M : m$; then, from the manner in which these teeth act upon each other, it appears that whilst the *smaller* wheel makes one revolution, the larger one will make only $\frac{m}{M}$th part of a revolution; and if (N) be the number of revolutions made by the *smaller* wheel in any given time, then $\frac{Nm}{M}$ will be the number of revolutions performed by the *larger* one in the *same time*.



2. Instead of *two* pairs of larger and smaller wheels between the *handle PQ* and *axis E,* as represented in this Figure, conceive now *any number* of small wheels *A, C, E, G,* &c. (each acting upon a corresponding *large* one, *B, D,*

B, D, F, H, &c.) to be interposed, and let the number of teeth in the *wheel* $A=m$, in the *wheel* $B=M$; in the *wheel* $C=m'$, and in the *wheel* $D=M'$; in the *wheel* $E=m''$, and in the *wheel* $F=M''$; &c. &c.; suppose also that the number of revolutions performed in a given time by the *wheel* $A=N$; by the *wheels* B or $C=N'$; by the *wheels* D or $E=N''$; by the *wheels* F or $G=N'''$; &c. then, from what has just been shewn, we shall have,

$$N' = \frac{Nm}{M}, \text{ and } \frac{M}{m} \ldots\ldots = \frac{N}{N'}.$$

$$N'' = \frac{N'm'}{M'} = \frac{Nmm'}{MM'}; \text{ and } \frac{M'}{m'} = \frac{N'}{N''}.$$

$$N''' = \frac{N''m''}{M''} = \frac{Nmm'm''}{MMM''}; \text{ and } \frac{M''}{m''} = \frac{N''}{N'''}.$$

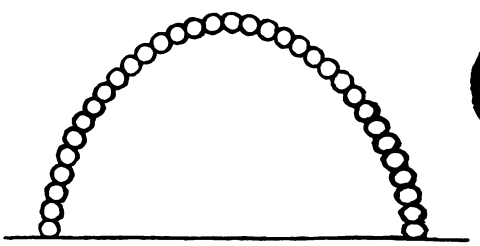
&c. = &c. = &c.

from which equations, "if the number of revolutions (N) "performed by the first wheel (A) in a given time and the "*number of teeth* in the several wheels be given, we can find "the *number of revolutions* performed by the wheels B, D, "F, &c. in the *same given time*;" and on the other hand, "if the number of revolutions performed by the several "wheels B, D, F, &c. in a given time be given, we can find the value of $\frac{M}{m}, \frac{M'}{m'}, \frac{M''}{m''}$, &c. or, *the proportion between the number of teeth in each corresponding larger and smaller wheel*;" moreover, since the teeth in each corresponding pair of wheels are supposed to be *equal*, this proportion will evidently be that of the *circumferences*, and consequently of the *radii* of the wheels B, A; D, C; F, E; &c. &c. This Theorem applies to the construction of wheels for regulating the motions of the *hour, minute* and *second* hand of a clock.

IV.

At the end of Lecture X. it might have been observed, that if a number of very small equal spheres be placed upon each other

other in the manner represented in the annexed Figure, and the whole supposed to be in equilibrio in a vertical plane, then the figure which the spheres (thus placed contiguous to and pressing on each other) will assume, will be that of an *inverted catenary*. For it is evident that these spheres are under precisely the same circumstances with respect to their equilibrium, as the particles of the catenary in its state of suspension; gravity operates upon the spheres in the same manner as it operates on the particles of the catenary; and since the spheres may be considered as pressing against each other in the direction of the straight line which joins their respective centers, this pressure is a force of the same kind with that of the *tension* of the different parts of the cate-



nary, but acts in an *opposite direction*; upon the same principle, therefore, that each particle of the catenary, in its state of *suspension*, is kept *at rest* by *its gravity and tension*, each of these spheres will be kept at rest by *its gravity and pressure between the two adjoining spheres*. In this *inverted* state it is called the curve of *equilibration*; and in this latter sense it lays the foundation of the theory for the "*Construction of Arches*;" for an explanation of which theory the Reader is referred to such authors as have written specifically upon the subject, more especially to Dr. HUTTON[a].

(a) *Tracts on Mathematical and Philosophical Subjects, vol.* I. 1812.

